UNIT 1 : SETS

UNIT STRUCTURE

1.1 LEARNING OBJECTIVES

After going through this unit, you will be able to

- describe sets and their representations
- identify empty set, finite and infinite sets
- define subsets, super sets, power sets, universal set
- describe the use of Venn diagram for geometrical description of sets

- illustrate the set operations of union, intersection, difference and complement
- know the different algebraic laws of set-operations
- illustrate the application of sets in solving practical problems.

1.2 INTRODUCTION

One of the widely used concepts in present day Mathematics is the concept of Sets. It is considered the language of modern Mathematics. The whole structure of Pure or Abstract Mathematics is based on the concept of sets. German mathematician **Georg Cantor** (1845-1918) developed the theory of sets and subsequently many branches of modern Mathematics have been developed based on this theory. In this unit, preliminary concepts of sets, set operations and some ideas on its practical utility will be introduced.

1.3 SETS AND THEIR REPRESENTATION

A set is a collection of well-defined objects. By well-defined, it is meant that given a particular collection of objects as a set and a particular object, it must be possible to determine whether that particular object is a member of the set or not.

The objects forming a set may be of any sort– they may or may not have any common property. Let us consider the following collections :

i) the collection of the prime numbers less than 15 i.e., 2, 3, 5, 7, 11, 13
ii) the collection of 0, a, Sachin Tendulkar, the river Brahmaputra
iii) the collection of the beautiful cities of India
iv) the collection of great mathematicians.

Clearly the objects in the collections (i) and (ii) are well-defined. For example, 7 is a member of (i), but 20 is not a member of (i). Similarly, ‘a’ is a member of (ii), but M. S. Dhoni is not a member. So, the collections (i) and (ii) are sets. But the collections (iii) and (iv) are not sets, since the objects in these collections are not well-defined.

The objects forming a set are called **elements** or **members** of the set. Sets are usually denoted by capital letters A, B, C, ...; X, Y, Z, ..., etc., and the elements are denoted by small letters a, b, c, ...; x, y, z, ..., etc.

If 'a' is an element of a set A, then we write $a \in A$ which is read as 'a belongs to the set A' or in short, 'a belongs to A'. If 'a' is not an element of A, we write $a \notin A$ and we read as 'a does not belong to A'. For example, let A be the set of prime number less than 15.

Then $2 \in A$, $3 \in A$, $5 \in A$, $7 \in A$, $11 \in A$, $14 \in A$

$1 \notin A$, $4 \notin A$, $17 \notin A$, etc.

Representation of Sets : Sets are represented in the following two methods :

1. Roster or tabular method
2. Set-builder or Rule method

In the Roster method, the elements of a set are listed in any order, separated by commas and are enclosed within braces, For example,

A = {2, 3, 5, 7, 11, 13}

B = {0, a Sachin Tendulcar, the river Brahmaputra}

C = {1, 3, 5, 7, ...}

In the set C, the elements are all the odd natural numbers. We cannot list all the elements and hence the dots have been used showing that the list continues indefinitely.

In the Rule method, a variable x is used to represent the elements of a set, where the elements satisfy a definite property, say P(x). Symbolically, the set is denoted by {x : P(x)} or {x | p(x)}. For example,

A = {x : x is an odd natural number}

B = {x : $x^2 - 3x + 2 = 0$}, etc.

If we write these two sets in the Roster method, we get,

A = {1, 3, 5, ...}

B = {1, 2}

Some Standard Symbols for Sets and Numbers : The following standard symbols are used to represent different sets of numbers :

N = {1, 2, 3, 4, 5, ...}, the set of natural numbers

Z = {..., –3, –2, –1, 0, 1, 2, 3, ...}, the set of integers

Q = {x : x = $^p/_q$; p, q $\in$ Z, q $\neq$ 0}, the set of rational numbers

R = {x : x is a real number}, the set of real numbers

NOTE

1) It should be noted that the symbol ':' of '|' stands for the phrase 'such that'.
2) While writing a set in Roster method, only distinct elements are listed. For example, if A is the set of the letters of the word MATHEMATICS, then we write
   A = {A, E, C, M, H, T, S, I}
   The elements may be listed in any order.

Z^+, Q^+, R^+ respectively represent the sets of positive integers, positive rational numbers and positive real numbers. Similarly Z^-, Q^-, R^- represent respectively the sets of negative integers, negative rational numbers and negative real numbers. Z^0, Q^0, R^0 represent the sets of non-zero integers, non-zero rational numbers and non-zero real numbers.

Illustrative Examples :

1. Examine which of the following collections are sets and which are not :

   i) the vowels of the English alphabet

   ii) the divisors of 56

   iii) the brilliant students degree-course of Guwahati

   iv) the renowned cricketers of Assam.

Solution :

i) It is a set, V = {a, e, i, o, u}

ii) It is a set, D = {1, 2, 4, 7, 8, 14, 28, 56}

iii) not a set, elements are not well-defined.

iv) not a set, elements are not well-defined.

2. Write the following sets in Roster method :

   i) the set of even natural numbers less than 10

   ii) the set of the roots of the equation $x^2–5x+6 = 0$

   iii) the set of the letters of the word EXAMINATION

Solution :

i) {2, 4, 6, 8}

ii) {2, 3}

iii) {E, X, A, M, I, N, T, O}

3. Write the following sets in Rule method :

   i) E = {2, 4, 6, ...}

   ii) A = {2, 4, 8, 16, 32}

   iii) B = {1, 8, 27, 64, 125, 216}

Solution :

i) $E = \{x : x = 2n, n \in N\}$

ii) $A = \{x : x = 2^n, n \in N, n < 6\}$

iii) $B = \{x : x = n^3, n \in N, n \leq 6\}$

CHECK YOUR PROGRESS

Q.1. Express the following sets in Roster method :

i) A = {x : x is a day of the week}

ii) B = {x : x is a month of the year}

iii) C = {x : $x^3 - 1 = 0$}

iv) D = {x : x is a positive divisor of 100}

v) E = {x : x is a letter of the word ALGEBRA}

Q.2. Express the following sets in Set-builder method :

i) A = {January, March, May, July, August, October, December}

ii) B = {0, 3, 8, 15, 24, ...}

iii) C = {0, ±5, ±10, ±15, ...}

iv) D = {a, b, c, ..., x, y, z}

Q.3. Write true or false :

i) $5 \in N$ ii) $\frac{1}{2} \in Z$ iii) $-1 \in Q$

iv) $\sqrt{2} \in R$ v) $\sqrt{-1} \in R$ vi) $-3 \notin N$

1.4 THE EMPTY SET

Definition : A set which does not contain any element is called an **empty set** or a **null set** or a **void set**. It is denoted by 'ϕ'.

The following sets are some examples of empty sets.

i) the set {x : $x^2 = 3$ and $x \in Q$}

ii) the set of people in Assam who are older than 500 years

iii) the set of real roots of the equation $x^2 + 4 = 0$

iv) the set of Lady President of India born in Assam.

1.5 FINITE AND INFINITE SETS

Let us consider the sets

A = {1, 2, 3, 4, 5}

and B = {1, 4, 7, 10, 13, ...}

If we count the members (all distinct) of these sets, then the counting process comes to an end for the elements of set A, whereas for the elements of B, the counting process does not come to an end. In the first case we say that A is a finite set and in the second case, B is called an infinite set. A has finite number of elements and number of elements in B are infinite.

Definition : A set containing finite number of distinct elements so that the process of counting the elements comes to an end after a definite stage is called a **finite set**; otherwise, a set is called an **infinite set**.

NOTE
A finite set can always be expressed in roster method. But an infinite set cannot be always expressed in roster method as the elements may not follow a definite pattern. For example, the set of real numbers, R cannot be expressed in roster method.

Example : State which of the following sets are finite and which are infinite.

i) the set of natural numbers N
ii) the set of male persons of Assam as on January 1, 2009.
iii) the set of prime numbers less than 20
iv) the set of concentric circles in a plane
v) the set of rivers on the earth.

Solution :

i) N = {1, 2, 3, ...} is an infinite set
ii) it is a finite set
iii) {2, 3, 5, 7, 11, 13, 17, 19} is a finite set
iv) it is an infinite set
v) it is a finite set.

1.6 EQUAL SETS

Definition : Two sets A and B are said to be equal sets if every element of A is an element of B and every element of B is also an element of A. In otherwords, A is equal to B, denoted by A = B if A and B have exactly the same elements. If A and B are not equal, we write $A \neq B$.

Let us consider the sets

$A = \{1, 2\}$

$B = \{x : (x-1)(x-2) = 0\}$

$C = \{x : (x-1)(x-2)(x-3) = 0\}$

Clearly $B = \{1, 2\}$, $C = \{1, 2, 3\}$ and hence $A = B$, $A \neq C$, $B \neq C$.

Example : Find the equal and unequal sets :

i) A = {1, 4, 9}

ii) B = {1^2, 2^2, 3^3}

iii) C = {x : x is a letter of the word TEAM}

iv) D = {x : x is a letter of the word MEAT}

v) E = {1, {4}, 9}

Solution : A = B, C = D, A $\neq$ C, A $\neq$ D, A $\neq$ E, B $\neq$ C, B $\neq$ D, B $\neq$ E, C $\neq$ E, D $\neq$ E

NOTE

According to equality of sets discussed above, the sets
A = {1, 2, 3} and
B = {1, 2, 2, 2, 3, 1, 3}
are equal, since every member of A is a member of B and also every member of B is a member of A. This is why identical elements are taken once only while writing a set in the Roster method.

1.7 SUBSETS, SUPERSETS, PROPER SUBSETS

Let us consider the sets A = {1, 2, 3}, B = {1, 2, 3, 4} and C = {3, 2, 1}. Clearly, every element of A is an element of B, but A is not equal to B. Again, every element of A is an element of C, and also A is equal to C. In both cases, we say that A is a subset of B and C. In particular, we say that A is a proper subset of B, but A is not a proper subset of C.

Definition : If every element of a set A is also an element of another set B, then A is called a **subset** of B, or A is said to be contained in B, and is denoted by A $\subseteq$ B. Equivalently, we say that B contains A or B is a **superset** of A and is denoted by B $\supseteq$ A. Symbolically, A $\subseteq$ B means that for all x, if x $\in$ A then x $\in$ B.

If A is a subset of B, but there exists atleast one element in B which is not in A, then A is called a **proper subset** of B, denoted by A $\subset$ B. In otherwords, A $\subset$ B $\Leftrightarrow$ (A $\subseteq$ B and A $\neq$ B).

The symbol '$\Leftrightarrow$' stands for 'logically implies and is implied by' (see unit 5).

Some examples of proper subsets are as follows :

N $\subset$ Z, N $\subset$ Q, N $\subset$ R,

Z $\subset$ Q, Z $\subset$ R, Q $\subset$ R.

It should be noted that any set A is a subset of itself, that is, A $\subseteq$ A. Also, the null set ϕ is a subset of every set, that is, $\phi \subseteq$ A for any set A. Because, if $\phi \subseteq$ A, then there must exist an element x $\in \phi$ such that x $\notin$ A. But x $\notin \phi$, hence we must accept that $\phi \subseteq$ A.

Combining the definitions of equality of sets and that of subsets, we get $A = B \Leftrightarrow (A \subseteq B \text{ and } B \subseteq A)$

Illustrative Examples :

1. Write true or false :

i) $1 \subset \{1, 2, 3\}$

ii) $\{1, 2\} \subseteq \{1, 2, 3\}$

iii) $\phi \subseteq \{\{\phi\}\}$

iv) $\phi \subseteq \{\phi, \{1\}, \{a\}\}$

v) $\{a, \{b\}, c, d\} \subset \{a, b, \{c\}, d\}$

Solution :

i) False, since $1 \in \{1, 2, 3\}$.

ii) True, since every element of $\{1, 2\}$ is an element of $\{1, 2, 3\}$.

iii) False, since ϕ is not an element of $\{\{\phi\}\}$.

iv) True, since ϕ is subset of every set.

v) False, since $\{b\} \notin \{a, b, \{c\}, d\}$ and $c \notin \{a, b, \{c\}, d\}$.

1.8 POWER SET

Let us consider a set $A = \{a, b\}$. A question automatically comes to our mind– 'What are the subsets of A?' The subsets of A are ϕ, $\{a\}$, $\{b\}$ and A itself.

These subsets, taken as elements, again form a set. Such a set is called the power set of the given set A.

Definition : The set consisting of all the subsets of a given set A as its elements, is called the **power set** of A and is denoted by P(A) or 2^A.

Thus, P(A) or $2^A = \{X : X \subseteq A\}$

Clearly,

i) $P(\phi) = \{\phi\}$

ii) if $A = \{1\}$, then $P^A = \{\phi, \{1\}\}$

iii) if $A = \{1, 2\}$, then $P^A = \{\phi, \{1\}, \{2\}, A\}$

iv) if $A = \{1, 2, 3\}$, then $P^A = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A\}$

From these examples we can conclude that if a set A has n elements, then P(A) has 2^n elements.

1.9 UNIVERSAL SET

A set is called a **Universal Set** or the **Universal discourse** if it contains all the sets under consideration in a particular discussion. A universal set is denoted by U.

Example :

i) For the sets {1, 2, 3}, {3, 7, 8}, {4, 5, 6, 9}

We can take U = {1, 2, 3, 4, 5, 6, 7, 8, 9}

ii) In connection with the sets N, Z, Q we can take R as the universal set.

iii) In connection with the population census in India, the set of all people in India is the universal set, etc.

CHECK YOUR PROGRESS

Q.4. Find the empty sets, finite and infinite sets :

i) the set of numbers divisible by zero

ii) the set of positive integers less than 15 and divisible by 17

iii) the set of planets of the solar system

iv) the set of positive integers divisible by 4

v) the set of coplaner triangles

vi) the set of Olympians from Assam participating in 2016, Rio Olympic.

Q.5. Examine the equality of the following sets :

i) A = {2, 3}, B = $\{x : x^2–5x+6 = 0\}$

ii) A = {x : x is a letter of the word WOLF}

B = {x : x is a letter of the word FLOW}

iii) A = {a, b, c}, B = {a, {b, c}}

Q.6. Write true or false :

i) $\{1, 3, 5\} \subseteq \{5, 1, 3\}$ ii) $\{a\} \subset \{\{a\}, b\}$

iii) $\{x : (x–1)(x–2) = 0\} \not\subset \{x : (x^2–3x+2)(x–3) = 0\}$

Q.7. Write down the power sets of the following sets :

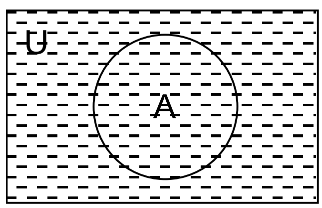
i) $A = \{1, 2, 3, 4\}$ ii) $B = \{1, \{2, 3\}\}$

Q.8. Give examples to show that $(A \subseteq B \text{ and } B \subseteq C) \Rightarrow A \subseteq C$.

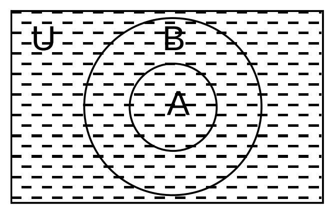
1.10 VENN DIAGRAM

Simple plane geometrical areas are used to represent relationships between sets in meaningful and illustrative ways. These diagrams are called **Venn-Euler** diagrams, or simply the **Venn-diagrams**.

In Venn diagrams, the universal set U is generally represented by a set of points in a rectangular area and the subsets are represented by circular regions within the rectangle, or by any closed curve within the rectangle. As an illustration Venn diagrams of $A \subset U$, $A \subset B \subset U$ are given below :



$A \subset U$



$A \subset B \subset U$

Similar Venn diagrams will be used in subsequent discussions illustrating different algebraic operations on sets.

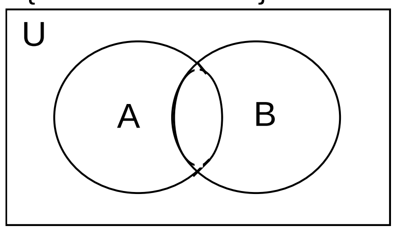
1.11 SET OPERATIONS

We know that given a pair of numbers x and y, we can get new numbers $x + y$, $x - y$, xy, $^{x}/_{y}$ (with $y \neq 0$) under the operations of addition, subtraction, multiplication and division. Similarly, given the two sets A and B we can form new sets under set operations of **union**, **intersection**, **difference** and **complements**. We will now define these set operations, and the new sets thus obtained will be shown with the help of Venn diagrams.

1.11.1 Union of Sets

Definition : The union of two sets A and B is the set of all elements which are members of set A or set B or both. It is denoted by $A \cup B$, read as 'A union B' where '$\cup$' is the symbol for the operation of 'union'. Symbolically we can describe $A \cup B$ as follows :

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$



$A \cup B$ (Shaded)

It is obvious that $A \subseteq A \cup B$, $B \subseteq A \cup B$

Example 1 : Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 5, 6\}$

Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$

Example 2 : Let Q be the set of all rational numbers and K be the set of all irrational numbers and R be the set of all real numbers.

Then $Q \cup K = R$

Identities : If A, B, C be any three sets, then

i) $A \cup B = B \cup A$

ii) $A \cup A = A$

iii) $A \cup \phi = A$

iv) $A \cup U = U$

v) $(A \cup B) \cup C = A \cup (B \cup C)$

Proof :

i) $A \cup B = \{x : x \in A \text{ or } x \in B\}$
$= \{x : x \in B \text{ or } x \in A\}$
$= B \cup A$

ii) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$

iii) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\} = A$

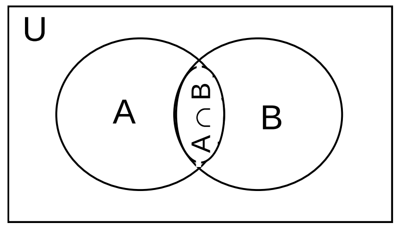
iv) $A \cup U = \{x : x \in A \text{ or } x \in U\}$
$= \{x : x \in U\}$, since $A \subset U$
$= U$

v) $(A \cup B) \cup C = \{x : x \in A \cup B \text{ or } x \in C\}$
$= \{x : (x \in A \text{ or } x \in B) \text{ or } x \in C\}$
$= \{x : x \in A \text{ or } (x \in B \text{ or } x \in C\}$
$= \{x : x \in A \text{ or } x \in B \cup C\}$
$= A \cup (B \cup C)$

1.11.2 Intersection of Sets

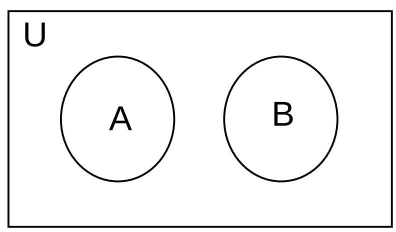
Definition : The intersection of two sets A and B is the set of all elements which are members of both A and B. It is denoted by $A \cap B$, read as 'A intersections B', where '$\cap$' is the symbol for the operation of 'intersection'. Symbolically we can describe it as follows:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$



$A \cap B$ (Shaded)

From definition it is clear that if A and B have no common element, then $A \cap B = \phi$. In this case, the two sets A and B are called **disjoint sets**.



$A \cap B = \phi$

It is obvious that $A \cap B \subseteq A$, $A \cap B \subseteq B$.

Example 1 : Let A = {a, b, c, d}, B = {b, d, 4, 5}

Then $A \cap B = \{b, d\}$

Example 2 : Let A = {1, 2, 3}, B = {4, 5, 6}

Then $A \cap B = \phi$.

Identities :

i) $A \cap B = B \cap A$

ii) $A \cap A = A$

iii) $A \cap \phi = \phi$

iv) $A \cap U = A$

v) $(A \cap B) \cap C = A \cap (B \cap C)$

vi) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof :

i) $A \cap B = \{x : x \in A \text{ and } x \in B\}$

$= \{x : x \in B \text{ and } x \in A\}$

$= B \cap A$

ii) $A \cap A = \{x : x \in A \text{ and } x \in A\}$

$= \{x : x \in A\}$

$= A$

iii) Since ϕ has no element, so A and ϕ have no common element.

Hence $A \cap \phi = \phi$

iv) $A \cap U = \{x : x \in A \text{ and } x \in U\}$

$= \{x : x \in A\}$, since $A \subset U$

$= A$

v) $(A \cap B) \cap C = \{x : x \in A \cap B \text{ and } x \in C\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ and } x \in C\}$

$= \{x : x \in A \text{ and } (x \in B \text{ and } x \in C\}$

$= \{x : x \in A \text{ and } x \in B \cap C\}$

$= A \cap (B \cap C)$

vi) $x \in A \cap (B \cup C) \Leftrightarrow x \in A \text{ and } x \in (B \cup C)$

$\Leftrightarrow x \in A \text{ and } (x \in B \text{ or } x \in C)$

$\Leftrightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$

$\Leftrightarrow x \in (A \cap B) \text{ or } x \in (A \cap C)$

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

So, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Similarly, it can be proved that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

NOTE

$x \in A \cap B$

$\Rightarrow x \in A \text{ and } x \in B$

But, $x \notin A \cap B$

$\Rightarrow x \notin A \text{ or } x \notin B$

Again, $x \in A \cup B$

$\Rightarrow x \in A \text{ or } x \in B$

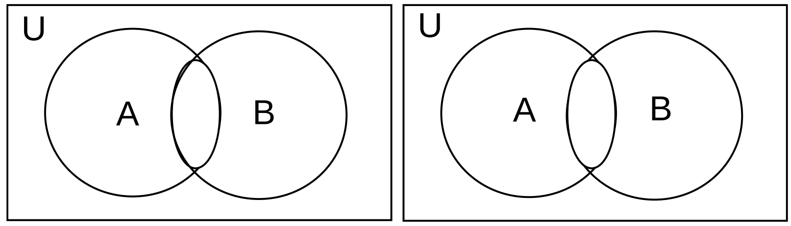
But, $x \notin A \cup B$

$\Rightarrow x \notin A \text{ and } x \notin B$

1.11.3 Difference of Sets

Definition : The difference of two sets A and B is the set of all elements which are members of A, but not of B. It is denoted by A–B. Symbolically, $A - B = \{x : x \in A \text{ and } x \notin B\}$

Similarly, $B - A = \{x : x \in B \text{ and } x \notin A\}$



A – B (Shaded) B – A (Shaded)

Example : Let A = {1, 2, 3, 4, 5}, B = {1, 4, 5}, C = {6, 7, 8}

Then $A - B = \{2, 3\}$

$A - C = A$

$B - C = B$

$B - A = \phi$

Properties :

i) $A - A = \phi$

ii) $A - B \subseteq A$, $B - A \subseteq B$

iii) $A - B$, $A \cap B$, $B - A$ are mutually disjoint and

$(A - B) \cup (A \cap B) \cup (B - A) = A \cup B$

iv) $A - (B \cup C) = (A - B) \cap (A - C)$

v) $A - (B \cap C) = (A - B) \cup (A - C)$

Proof : We prove (iv), others are left as exercises.

$$\begin{aligned} x \in A - (B \cup C) &\Leftrightarrow x \in A \text{ and } x \notin (B \cup C) \\ &\Leftrightarrow x \in A \text{ and } (x \notin B \text{ and } x \notin C) \\ &\Leftrightarrow (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C) \\ &\Leftrightarrow x \in (A - B) \text{ and } x \in (A - C) \\ &\Leftrightarrow x \in (A - B) \cap (A - C) \end{aligned}$$

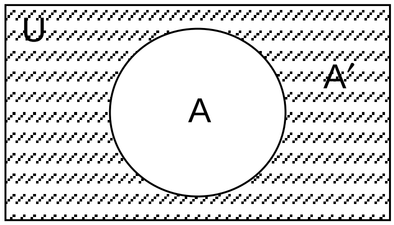
So, $A - (B \cup C) \subseteq (A - B) \cap (A - C)$, $(A - B) \cap (A - C) \subseteq A - (B \cup C)$

Hence, $A - (B \cup C) = (A - B) \cap (A - C)$.

1.11.4 Complement of a Set

Definition : If U be the universal set of a set A, then the set of all those elements in U which are not members of A is called the **Compliment** of A, denoted by A^C or A'.

Symbolically, $A' = \{x : x \in U \text{ and } x \notin A\}$.



A′ (Shaded)

Clearly, $A' = U - A$.

Example : Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{2, 4, 6, 8\}$

Then $A' = \{1, 3, 5, 7, 9\}$

Identities : i) $U' = \phi$, $\phi' = U$

ii) $(A')' = A$

iii) $A \cup A' = U$, $A \cap A' = \phi$

iv) $A - B = A \cap B'$, $B - A = B \cap A'$

v) $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$

> **NOTE**
>
> The identities $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$ are know as **DeMorgan's Laws**.

Proof : We prove $(A \cup B)' = A' \cap B'$. The rest are left as exercises.

$$
\begin{aligned}
(A \cup B)' &= \{x : x \in U \text{ and } x \notin A \cup B\} \\
&= \{x : x \in U \text{ and } (x \notin A \text{ and } x \notin B\} \\
&= \{x : (x \in U \text{ and } x \notin A) \text{ and } (x \in U \text{ and } x \notin B)\} \\
&= \{x : x \in A' \text{ and } x \in B'\} \\
&= A' \cap B'.
\end{aligned}
$$

Illustrative Examples :

1. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

   $A = \{2, 4, 6, 8, 10\}$

   $B = \{3, 6, 9\}$

   and $C = \{1, 2, 3, 4, 5\}$, then find

   (i) $A \cup B$, (ii) $A \cap C$, (iii) $B \cap C$, (iv) A', (v) $A \cup B'$, (vi) $C' \cap B$,

   (vii) $A' \cup C'$, (viii) $A - C$, (ix) $A - (B \cup C)'$, (x) $A' \cap B'$.

Solution : i) $A \cup B = (2, 3, 4, 6, 8, 9, 10\}$

ii) $A \cap C = \{2, 4\}$

iii) $B \cap C = \{3\}$

iv) $A' = \{1, 3, 5, 7, 9\}$

v) $B' = \{1, 2, 4, 5, 7, 8, 10\}$

So, $A \cup B' = \{1, 2, 4, 5, 6, 7, 8, 10\}$

vi) $C' = \{6, 7, 8, 9, 10\}$

So, $C' \cap B = \{6, 9\}$

vii) From (iv) & (vi), $A' \cup C' = \{1, 3, 5, 6, 7, 8, 9, 10\}$

viii) $A - C = \{6, 8, 10\}$

ix) $B \cup C = \{1, 2, 3, 4, 5, 6, 9\}$

$(B \cup C)' = \{7, 8, 10\}$

So, $A - (B \cup C)' = \{2, 4, 6\}$

x) From (iv) & (v), $A' \cap B' = \{1, 5, 7\}$.

2. Verify the identities :

i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ii) $(A \cup B)' = A' \cap B'$

taking $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{3, 4, 5\}$ and

$U = \{1, 2, 3, 4, 5, 6\}$.

Solution : i) $B \cap C = \{3, 4\}$

$A \cup (B \cap C) = \{1, 2, 3, 4\}$ ......(1)

$A \cup B = \{1, 2, 3, 4\}$, $A \cup C = \{1, 2, 3, 4, 5\}$

$(A \cup B) \cap (A \cup C) = \{1, 2, 3, 4\}$ ......(2)

From (1) & (2), we get

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

ii) $A' = \{4, 5, 6\}$, $B' = \{1, 5, 6\}$

$A' \cap B' = \{5, 6\}$ ......(3)

$(A \cup B)' = \{1, 2, 3, 4\}' = \{5, 6\}$ ......(4)

From (3) & (4), wet get

$(A \cup B)' = A' \cap B'$.

CHECK YOUR PROGRESS

Q.9. Find the following sets :

i) $\phi \cap \{\phi\}$ ii) $\{\phi\} \cap \{\phi\}$

iii) $\{\phi, \{\phi\}\} - \{\phi\}$ iv) $\{\phi, \{\phi\}\} - \{\{\phi\}\}$

Q.10. If $A = \{a, b, c\}$, $B = \{c, d, e\}$, $U = \{a, b, c, d, e, f\}$ then find

i) $A \cup B$ ii) $A \cap B$ iii) $A - B$ iv) $B - A$ v) A'

Q.11. If $A = \{3n : n \in N\}$, $B = \{n : n \in N, n < 20\}$ then find $A \cap B$, $B - A$.

Q.12. If $A = \{x : x \in R, 1 < x \leq 7\}$, $B = \{x : x \in R, 3 \leq x < 12\}$ then find $A \cup B$, $A \cap B$, $A - B$, $B - A$.

Q.13. Using numerical examples, virify that :

i) $A - B = B' - A'$

ii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap C)$

iii) $A - (B \cap C) = (A - B) \cup (A - C)$

Q.14. Prove that : i) $B - A \subseteq A'$ ii) $B - A' = B \cap A$

iii) $A \cup B = \phi \Rightarrow A = \phi$ and $B = \phi$.

1.12 LAWS OF THE ALGEBRA OF SETS

In the preceding discussions we have stated and proved various identities under the operations of union, intersection and complement of sets. These identities are considered as **Laws of Algebra of Sets**. These laws can be directly used to prove different propositions on Set Theory. These laws are given below :

1. **Idempotent laws :** $A \cup A = A$, $A \cap A = A$
2. **Commutative laws :** $A \cup B = B \cup A$, $A \cap B = B \cap A$
3. **Associative laws :** $A \cup (B \cup C) = (A \cup B) \cup C$,
   $A \cap (B \cap C) = (A \cap B) \cap C$
4. **Distributive laws :** $A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$,
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. **Identity laws :** $A \cup \phi = A$, $A \cup U = U$
   $A \cap U = A$, $A \cap \phi = \phi$
6. **Complement laws :** $A \cup A' = U$, $A \cap A' = \phi$
   $(A')' = A$, $U' = \phi$, $\phi' = U$
7. **De Morgan's laws :** $(A \cup B)' = A' \cap B'$
   $(A \cap B)' = A' \cup B'$.

Let us illustrate the application of the laws in the following examples :

Example 1 : Prove that $A \cap (A \cup B) = A$

Solution : $A \cap (A \cup B) = (A \cup \phi) \cap (A \cup B)$, using identity law

$= A \cup (\phi \cap B)$, using distributive law

$= A \cup (B \cap \phi)$, using commutative law

$= A \cup \phi$, using identity law

$= A$, again using identity law

Example 2 : Prove that $A \cap (A' \cup B) = A \cap B$

Solution : $A \cap (A' \cup B) = (A \cap A') \cup (A \cap B)$, using distributive law

$= \phi \cup (A \cap B)$, using complement law

$= (A \cap B) \cup \phi$, using commutative law

$= A \cap B$, using identity law

1.13 TOTAL NUMBER OF ELEMENTS IN UNION OF SETS IN TERMS OF ELEMENTS IN INDIVIDUAL SETS AND THEIR INTERSECTIONS

We shall now prove a theorem on the total number of elements in the union of two sets in terms of the number of elements of the two individual sets and the number of elements in their intersection. Its application in solving some practical problems concerning everyday life will be shown in the illustrative examples.

Theorem : If A and B are any two finite sets,

then $|A \cup B| = |A| + |B| - |A \cap B|$

[The symbol $|S|$ represents total number of elements in a set S]

Proof : Let $|A| = n$, $|B| = m$, $|A \cap B| = k$

Then from the Venn diagram,

we get $|A - B| = n - k$, $|B - A| = m - k$

We know that $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$

Where $A - B$, $A \cap B$, $B - A$ are mutually disjoint.

Hence $|A \cup B| = |A - B| + |A \cap B| + |B - A|$

$= (n - k) + k + (m - k)$

$= n + m - k$

$= |A| + |B| - |A \cap B|$

Deduction :

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

Proof : $|A \cup B \cup C| = |(A \cup B) \cup C|$

$$= |A \cup B| + |C| - |(A \cup B) \cap C|$$

$$= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)|$$

$$= |A| + |B| + |C| - |A \cap B| - [|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|]$$

$$= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Corollaries :

i) If A and B are disjoint, then $A \cap B = \phi$ and so $|A \cap B| = 0$.

Hence $|A \cup B| = |A| + |B|$,

which is known as the **Sum Rule of Counting**.

ii) If A, B and C are mutually disjoint, then as above

$|A \cup B \cup C| = |A| + |B| + |C|$.

Illustrative Examples :

1. In a class of 80 students, everybody can speak either English or Assamese or both. If 39 can speak English, 62 can speak Assamese, how many can speak both the languages?

Solution : Let A, B be the sets of students speaking English and Assamese respectively.

Then $|A \cup B| = 80$, $|A| = 39$, $|B| = 62$.

We are to find $|A \cap B|$.

Now $|A \cup B| = |A| + |B| - |A \cap B|$

So, $|A \cap B| = |A| + |B| - |A \cup B| = 39 + 62 - 80 = 21$.

Hence, 21 students can speak both the languages.

2. Among 60 students in a class, 28 got class I in SEM I and 31 got class I in SEM II. If 20 students did not get class I in either SEMESTERS, how many students got class I in both the SEMESTERS?

Solution : Let A and B be the sets of students who got class I in SEM I and SEM II respectively.

So, $|A| = 28$, $|B| = 31$.

20 students did not get class I in either SEMESTERS out of 60 students in the class.

Hence $|A \cup B| = 60 - 20 = 40$

But $|A \cup B| = |A| + |B| - |A \cap B|$

i.e., $40 = 28 + 31 - |A \cap B|$ So, $|A \cap B| = 19$

Therefore, 19 students did not get class I is both the SEMESTERS.

3. Out of 200 students, 70 play cricket, 60 play football, 25 play hockey, 30 play both cricket and football, 22 play both cricket and hockey, 17 play both football and hockey and 12 play all the three games. How many students do not play any one of the three games?

Solution : Let C, F, H be the sets of students playing cricket, football and hockey respectively. Then

$|C| = 70$, $|F| = 60$, $|H| = 25$,

$|C \cap F| = 30$, $|C \cap H| = 22$, $|F \cap H| = 17$, $|C \cap F \cap H| = 12$.

$$\begin{aligned}\text{So, } |C \cup F \cup H| &= |C| + |F| + |H| - |C \cap F| - |C \cap H| - |F \cap H| + |C \cap F \cap H| \\ &= 70 + 60 + 25 - 30 - 22 - 17 + 12 = 98\end{aligned}$$

Thus 98 students play atleast one of the three games.

Hence, number of students not playing any one of the three games $= 200 - 98 = 102$.

EXERCISES

1. Write ‘true’ or ‘false’ with proper justification :
   i) the set of even prime numbers is an empty set
   ii) $\{x : x+2 = 5, x < 0\}$ is an empty set
   iii) $\{3\} \subset \{1, 2, 3\}$
   iv) $x \in \{\{x, y\}\}$
   v) $\{a, b\} \subseteq \{a, b, \{c\}\}$
   vi) if A be any set, then $\phi \subseteq A \subseteq U$
2. Which of the following sets are equal?

   $A = \{x : x^2 + x - 2 = 0\}$

B $= \{x : x^2 - 3x + 2 = 0\}$

C $= \{x : x \in Z, |x| = 1\}$

D $= \{-2, 1\}$

E $= \{1, 2\}$

F $= \{x : x^2 - 1 = 0\}$

3. Find the sets which are finite and which are infinite :
   i) the set of natural numbers which are multiple of 7
   ii) the set of all districts of Assam
   iii) the set of real numbers between 0 and 1
   iv) the set of lions in the world.
4. If $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{0, 2, 3, 6\}$, $B = \{1, 2, 6, 8\}$, $C = \{3, 7, 8, 9\}$, then find A', B', C', $(A \cup B) \cap C$, $(A \cup B') \cup C'$, $(A \cap B) \cap C'$, $(A - C) \cup B'$, $(B - A)' \cap C$.
5. If $A \cup B = B$ and $A \cap B = B$, then what is the relation between A and B?
6. Verify the following identities with numerical examples :
   i) $A - B = B' - A'$
   ii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
   iii) $A - (B \cap C) = (A - B) \cup (A - C)$
   iv) $A - (B \cup C) = (A - B) \cap (A - C)$.
7. Write down the power set of the set $A = \{\{\phi\}, a, \{b, c\}\}$.
8. Given $A = \{\{a, b\}, \{c\}, \{d, e, f\}\}$, how many elements are there in $P(A)$?
9. Using numerical examples, show that
   i) $(A \cap B) \cup (A - B) = A$
   ii) $A \cup B = A \cup (B - A)$
   iii) $A \cup B = B \cup (A - B)$
   iv) $B - A \subseteq A'$
   v) $B - A' = B \cap A$.
10. Using Venn diagrams show that
    i) $A \cup B \subset A \cup C$ but $B \not\subset C$
    ii) $A \cap B \subset A \cap C$ but $B \not\subset C$
    iii) $A \cup B = A \cup C$ but $B \neq C$.

11. Give numerical examples for the results given in 10.
12. Show that (A – B) – C = (A – C) – (B – C).
13. Out of 100 persons, 45 drink tea and 35 drink coffee. If 10 persons drink both, how many drink neither tea nor coffee?
14. Using sets, find the total number of integers from 1 to 300 which are not divisible by 3, 5 and 7.
15. 90 students in a class appeared in tests for Physics, Chemistry and Mathematics. If 55 passed in Physics, 45 passed in Chemistry, 60 passed in Mathematics, 40 both in Physics and Chemistry, 30 both in Chemistry and Mathematics, 35 both in Physics and Mathematics and 20 passed in all the three subjects, then find the number of students failing in all the three subjects.

1.14 LET US SUM UP

- A set is a collection of well-defined and distinct objects. The objects are called members or elements of the set.
- Sets are represented by capital letters and elements by small letters. If 'a' is an element of set A, we write $a \in A$, otherwise $a \notin A$.
- Sets are represented by (i) Roster or Tabular method and (ii) Rule or Set-builder method.
- A set having no element is called **empty set** or **null set** or **void set**, denoted by ϕ.
- A set having a finite number of elements is called a **finite set**, otherwise it is called an **infinite set**.
- Two sets A and B are equal, i.e. A = B if and only if every element of A is an element of B and also every element of B is an element of A, otherwise $A \neq B$.
- A is a **subset** of B, denoted by $A \subseteq B$ if every element of A is an element of B and A is a **proper subset** of B if $A \subseteq B$ and $A \neq B$. In this case, we write $A \subset B$.
- A = B if and only if $A \subseteq B$ and $B \subseteq A$.

- The set of all the subsets 8 a set A is called the **power set** of A, denoted by P(A) or 2^A. If $|A| = n$, then $|P(A)| = 2^n$.
- Venn diagrams are plane geometrical diagrams used for representing relationships between sets.
- The union of two sets A and B is $A \cup B$ which consists of all elements which are either in A or B or in both. $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- The intersection of two sets A and B is $A \cap B$ which consists of all the elements common to both A and B.
- For any two sets A and B, the difference set, $A - B$ consists of all elements which are in A, but not in B. $A - B = \{x : x \in A \text{ and } x \notin B\}$
- The **Universal set U** is that set which contains all the sets under any particular discussion as its subsets.
- The **complement** of a set A, denoted by A^c or A' is that set which consists of all those elements in U which are not in A.

  $A' = \{x : x \in U \text{ and } x \notin A\} = U - A$
- Following are the **Laws of Algebra of Sets** :

  $A \cup A = A,\ A \cap A = A$

  $A \cup B = B \cup A,\ A \cap B = B \cap A$

  $A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap B) \cap C$

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C),\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

  $A \cup \phi = A,\ A \cup U = U,\ A \cap U = A,\ A \cap \phi = \phi$

  $A \cup A' = U,\ A \cap A' = \phi,\ (A')' = A,\ U' = \phi,\ \phi' = U$

  $(A \cup B)' = A' \cap B',\ (A \cap B)' = A' \cup B'$.
- $|A \cup B| = |A| + |B| - |A \cap B|$

  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

1.15 ANSWERS TO CHECK YOUR PROGRESS

Ans. to Q. No. 1 : i) A = {Monday, Tuesday, Wednessday, Thursday, Friday, Saturday, Sunday}

ii) B = {January, February, March, April, May, June, July, August, September, October, November, December}

iii) $C = \{1, w, w^2\}$

iv) $D = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$

v) $E = \{A, B, E, G, L, R\}$

Ans. to Q. No. 2 : i) $A = \{x : x \text{ is a month of the year having 31 days}\}$

ii) $B = \{x : x = n^2 - 1, n \in N\}$

iii) $C = \{x : x = 5n, n \in Z\}$

iv) $D = \{x : x \text{ is a letter of the English Alphabet}\}$

Ans. to Q. No. 3 : i) True, ii) False, iii) True, iv) True, v) False, vi) True.

Ans. to Q. No. 4 : i) ϕ, ii) ϕ, iii) finite, iv) infinite, v) infinite, vi) ϕ.

Ans. to Q. No. 5 : i) $B = \{2, 3\} = A$

ii) $A = \{W, O, L, F\}$, $B = \{F, L, O, W\}$ and so, $A = B$

iii) $A \neq B$; since $b \in A$ but $b \notin B$.

Ans. to Q. No. 6 : i) True

ii) False, since $\{a\} \in \{\{a\}, b\}$

iii) $\{x : (x-1)(x-2) = 0\} = \{1, 2\}$, $\{x : (x^2-3x+2)(x-3) = 0\} = \{1, 2, 3\}$

Hence $\{x : (x-1)(x-2) = 0 \subset \{x : (x^2-3x+2)(x-3) = 0\}$ and so, the given result is false.

Ans. to Q. No. 7 : i) $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\},$
$\{2, 3\}, \{2, 4\}, \{3, 4\},$
$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, A\}$

ii) $P(B) = \{\phi, \{1\}, \{\{2, 3\}\}, B\}$

Ans. to Q. No. 8 : Let $A = \{1, 2\}$, $B = \{0, 1, 2, 3\}$, $C = \{0, 1, 2, 3, 4, 5, 7\}$

Ans. to Q. No. 9 : i) ϕ, ii) $\{\phi\}$, iii) $\{\{\phi\}\}$, iv) $\{\phi\}$

Ans. to Q. No. 10 : i) $A \cap B = \{a, b, c, d, e\}$; ii) $\{C\}$; iii) $A - B = \{a, b\}$,
iv) $B - A = \{d, e\}$; v) $A' = \{d, e, f\}$

Ans. to Q. No. 11 : $A = \{3, 6, 9, 12, 15, 18, 21, ...\}$,
$B = \{1, 2, 3, ..., 18, 19, 20\}$,
Hence $A \cap B = \{3, 6, 9, 12, 15, 18\}$
and $B - A = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19\}$

Ans. to Q. No. 12 : $A \cup B = \{x : 1 < x < 12, x \in R\}$
$A \cap B = \{x : x \in R, 3 \leq x \leq 7\}$
$A - B = \{x : x \in R, 1 < x < 3\}$
$B - A = \{x : x \in R, 7 < x < 12\}$

Ans. to Q. No. 13 : Take $U = \{p, q, r, s, t, u, v, w, x, y, z\}$

$A = \{p, q, u, v, x, y\}$

$B = \{q, v, y, z\}$ and $C = \{p, s, t, v, x, y\}$

i) $A - B = \{p, u, x\}$, $A' = \{r, s, t, w, z\}$, $B' = \{p, r, s, t, u, x\}$

$B' - A' = \{p, u, x\}$ and hence, $A - B = B' - A'$

ii) $B - A = \{z\}$ and so, $(A - B) \cup (B - A) = \{p, u, x, z\}$

Again, $A \cup B = \{p, q, u, v, x, y, z\}$ and $A \cap B = \{q, v, y\}$

So, $(A \cup B) - (A \cap B) = \{p, u, x, z\}$

Thus, $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

iii) $A - C = \{q, u\}$ and so, $(A - B) \cup (A - C) = \{p, q, u, x\}$

$B \cap C = \{v, y\}$ and so, $A - (B \cap C) = \{p, q, u, x\}$

Thus $A - (B \cap C) = (A - B) \cup (A - C)$.

Ans. to Q. No. 14 : i) $x \in B - A \Rightarrow x \in B$ and $x \notin A \Rightarrow x \in U$ and $x \notin A$

$\Rightarrow x \in A'$,

Where x is an arbitrary element of $(B - A)$. Hence $B - A \subseteq A'$

ii) $x \in B - A' \Leftrightarrow x \in B$ and $x \notin A'$

$\Leftrightarrow x \in B$ and $x \in A \Leftrightarrow x \in B \cap A$

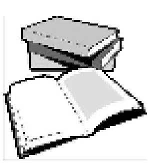
Hence $B - A' \subseteq B \cap A$ and $B \cap A \subseteq B - A'$

Thus, $B - A' = B \cap A$

iii) $A \subseteq A \cup B \Rightarrow A \subseteq \phi$, as $A \cup B = \phi$ ......(1)

Also $\phi \subseteq A$ ......(2)

From (1) & (2), we get $A = \phi$. Similarly, $B = \phi$.



1.16 FURTHER READINGS

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1.17 MODEL QUESTIONS

Q.1. Give examples of i) five null sets

ii) five finite sets

iii) five infinite sets

Q.2. Write down the following sets in rule method :

i) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$

ii) $B = \left\{\frac{1}{1.2}, \frac{1}{1.3}, \frac{1}{3.4}, \frac{1}{4.5}, \ldots\right\}$

iii) C = {2, 5, 10, 17, 26, 37, 50}

Q.3. Write down the following sets in roster method

i) $A = \{x : x \in N, 2 < x < 10\}$

ii) $B = \{x : x \in N, 4+x < 15\}$

iii) $C = \{x : x \in Z, -5 \le x \le 5\}$

Q.4. If A = {1, 3}, B = {1, 3, 5, 9}, C = {2, 4, 6, 8} and

D = {1, 3, 5, 7, 9} then fill up the dots by the symbol $\subseteq$ or $\not\subseteq$:

i) A ... B, ii) A ... C, iii) C ... D, iv) B ... D

Q.5. Write true or false :

i) $4 \in \{1, 2, \{3, 4\}, 5\}$, (ii) $\phi = \{\phi\}$

iii) A = {2, 3} is a proper subset of $B = \{x : (x-1)(x-2)(x-3) = 0\}$

iv) $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$

Q.6. If $U = \{x : x \in N\}$, $A = \{x : x \in N, x \text{ is even}\}$, $B = \{x : x \in N, x < 10\}$

$C = \{x : x \in N, x \text{ is divisible by } 3\}$, then find

i) $A \cup B$, ii) $A \cap C$, iii) $B \cap C$, iv) A', (v) B', vi) C'.

Q.7. If $A \cup B = B$ and $B \cup C = C$, then show that $A \subseteq C$.

Q.8. If U = {–5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5},

A = {–5, –2, 1, 2, 4}

B = {–2, –3, 0, 2, 4, 5}

C = {1, 0, 2, 3, 4, 5},

then find i) $A \cup B$, ii) $A \cap C$, iii) $A \cap (B \cup C)$, iv) $B \cap C'$,

v) $A' \cup (B \cap C')$, vi) $A - C'$, vii) $A - (B \cup C)'$, viii) $(B \cup C')$,

ix) $A' \cup C'$, x) $(C' \cup B) - A$.

Q.9. Prove the following :

i) If A, B, C are three sets such that $A \subseteq B$,

then $A \cup C \subseteq B \cup C$, $A \cap C \subseteq B \cap C$.

ii) $A \subseteq B$ if and only if $B' \subseteq A'$.

iii) $A \subseteq B$ if and only if $A \cap B = A$.

iv) If $A \cap B = \phi$, then $A \subseteq B'$.

Q.10. How many elements are there in P(A) if A has

i) 5 elements, ii) 2^n elements?

Q.11. Every resident in Guwahati can speak Assamese or English or both. If 80% can speak Assamese and 30% can speak both the language, what percent of residents can speak English?

Q.12. 76% of the students of a college drink tea and 63% drink coffee. Show that a minimum of 39% and a minimum of 63% drink both tea and coffee.

Q.13. In a survey of 100 students it is found that 40 read Readers' Digest, 32 read India Today, 26 read the Outlook, 10 read both Readers' Digest and India Today, 6 read India Today and the Outlook, 7 read Readers' Digest and the Outlook and 5 read all the three. How many read none of the magazines?

Q.14. In an examination 60% students passed in Mathematics, 50% passed in Physics, 40% passed in Computer Science, 20% passed in both Mathematics and Physics, 40% passed in both Physics and Computer Science, 30% passed in both Mathematics and Computer Science and 10% passed in all the three subjects. What percent failed in all the three subjects?

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UNIT2: RELATIONS AND FUNCTIONS

UNIT STRUCTURE

2.1 LEARNING OBJECTIVES

After going through this unit, you will be able to know:

- the concept of a relation
- different types of relation
- equivalence relation, equivalence class
- the concept of a function
- different types of function.

2.2 INTRODUCTION

Set theory may be called the language of modern mathematics. We know that a set is a well-defined collection of objects. Also we know the notion of subset of a set. If every element of a set B is in set A, then B is a subset of A. Symbolically we denote it by $B \subseteq A$.

Also we note that if A is a finite set having n elements, the number of subsets of A is 2^n.

Again if A, B are two non-empty sets, the cartesian product of A and B is denoted by A x B and is defined by $A \times B = \{(a, b) : a \in A, b \in B\}$

(a, b) is called an ordered pair.

Let $a \in A$, $c \in A$; $b \in B$, $d \in B$.

$\therefore$ $(a, b) \in A \times B$, $(c, d) \in A \times B$

We know that $(a, b) = (c, d) \Leftrightarrow a=c, b=d$.

Also we know that if A, B are finite and n(A)=x, n(B)= y, then

$$n(AxB) = n(BxA)=xy$$

[Here n(A) denotes the number of elements of A]

If one of the sets A and B is infinite, then A x B and B x A are infinite.

In this unit we will study relations and functions which are subsets of cartesian product of two sets.

We will denote the set of natural numbers by 'N', the set of integers by Z, the set of rational numbers by Q, the set of real numbers by IR, the set of complex numbers by C.

2.3 CONCEPT OF RELATION

Let us consider the following sentences.

(1) 11 is greater then 10.

(2) 35 is divisible by 7.

(3) New Delhi is the capital of India.

In each of the sentences there is a relation between two 'objects'.

Now let us see what is meant by relation in set theory.

Definition Let A and B be two non-empty sets. A subset R of A x B is said to be a **relation** from A to B.

If A=B, then any subset of A x A is said to be a relation on A.

If $R \subseteq A \times B$, and $(a, b) \in R$; $a \in A$, $b \in B$, it is also written as aRb and is read as 'a is R related to b'.

Note 1. The set of the first components of the ordered pairs of R is called the domain and the set of the second components of the ordered pairs of R is called the range of R.

2. If A, B are finite sets and n(A)=x, n(B)=y; then n(AxB)=xy. So, the number of subsets of AxB is 2^{xy}.

Therefore, the number of relations from A to B is 2^{xy}.

Example 1: Let A ={1, 2, 3},

B={8, 9}

$\therefore$ A x B ={(1, 8), (1, 9), (2, 8), (2, 9), (3, 8), (3, 9)}

Let R = {(1, 8), (2, 9), (3, 9)}

Clearly $R \subseteq AxB$

$\therefore$ R is a relation from A to B.

Here 1R8, 2R9, 3R9

Domain of R = {1, 2, 3}

Range of R = {8, 9}

Example 2 : Let IR be the set of real numbers.

Let R = {(x, y) : x, y $\in$ IR, x<y} $\subseteq$ IR x IR

$\therefore$ R is a relation of IR.

$\because$ 3<5, $\therefore$ (3, 5)$\in$R i.e. 3R5

19<27, $\therefore$ (19, 27)$\in$R i.e. 19R27

But 5>3, $\therefore$ (5, 3)$\notin$ R, i.e. $5 \not{R} 3$.

Example 3: Let X be the set of odd integers.

Let R = {(x, y) : x, y $\in$ X and x+y is odd}

We know that the sum of two odd integers is an even integer.

$\therefore$ if x, y are odd, then x+y cannot be odd.

$\therefore$ R = $\phi \subseteq$ X x X

In this case R is called a null relation on X.

Example 4: Let E be the set of even integers.

Let R = {(x, y) : x, y $\in$ E and x+y is even}

We know that the sum of the even integers is an even integer.

$\therefore$ if x, y are even, then x+y is always even.

$\therefore$ R = E x E $\subseteq$ E x E

In this case R is called a universal relation on E.

2.3.1 Identity Relation

Let A be a non-empty set.

$I_A = \{(a, a) : a \in A\} \subseteq A \times A$

I_A is called the identity relation on A.

Example 5: Let A = {1, 2, 3, 4}

Then $I_A = \{(1, 1), (2, 2), (3, 3), (4, 4)\} \subseteq A \times A$.

Clearly I_A is the identity relation on A.

2.3.2 Inverse Relation

Let A, B be two non-empty sets. Let R be a relation from A to B, i.e. $R \subseteq A \times B$. The inverse relation of R is denoted by R^{-1}, and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\} \subseteq B \times A$

Clearly, domain of R^{-1} = range of R

range of R^{-1} = domain of R

Example 6: Let A = {1, 2, 3}

B = {5, 6}

$R = \{(1, 5), (2, 6), (3, 5)\} \subseteq A \times B$

$R^{-1} = \{(5, 1), (6, 2), (5, 3)\} \subseteq B \times A$.

CHECK YOUR PROGRESS 1

Q 1: Let A, B be two finite sets, and n(A)=4, n(B)=3. How many relations are there from A to B?

Q 2: Let A be a finite set such that n(A) =5.
Write down the number of relations on A.

Q 3: Let IN be the set of natural numbers.
Let $R = \{(x, y) : x, y \in IN, x>y\}$.
Examine if the following ordered pairs belong to R.
(7, 4), (23, 32), (5, 5), (72, 27), (18, 19).

Q 4: Let E be the set of even integers.
Let $R = \{(x, y) : x, y \in E$ and $x+y$ is odd$\}$
Is R a null relation on E ? Justify your answer.

Q 5: Let A = {1, 2, 3, 4, 5}

Write down the identity relation on A.

Q 6: Let A = {4, 5, 6}, B = {7, 8, 9}

Let R = {(4, 7), (5, 8), (6, 7), (6, 8), (6, 9)}

Write down R^{-1}.

2.4 TYPES OF RELATION

Let A be a non-empty set, and R be a relation on A, i.e. $R \subseteq A \times A$.

1. R is called reflexive if $(a, a) \in R$, i.e; aRa, for all $a \in A$.
2. R is called symmetric if whenever $(a, b) \in R$, then $(b, a) \in R$, i.e, if whenever aRb, then bRa; $a, b \in A$.
3. R is called anti-symmetric if $(a, b) \in R$, $(b, a) \in R \Rightarrow a = b$, i.e., if aRb, bRa $\Rightarrow$ a=b; $a, b \in A$.
4. R is called transitive if whenever (a, b), $(b, c) \in R$, then $(a, c) \in R$, i.e., if whenever aRb, bRc, then aRc; $a, b, c \in A$.

Example 7: Let A = {1, 2, 3}

R = (1, 1), (2, 2), (1, 2), (2, 1)}

Examine if R is reflexive, symmetric,anti-symmetric, transitive.

Solution: Here $(1, 1) \in R$, $(2, 2) \in R$; but $(3, 3) \notin R$.

\ R is not reflexive.

Again, $(a, b) \in R \Rightarrow (b, a) \in R$

\ R is symmetric.

Again, $(1, 2) \in R$, $(2, 1) \in R$, but $1 \neq 2$

\ R is not anti-symmetric.

Again, if $(a, b) \in R$, $(b, c) \in R$, then $(a, c) \in R$

\ R is transitive.

Example 8: Let Z be the set of integers, and R = $\{(x, y) : x, y \in Z, x \leq y\}$

Examine if R is reflexive, symmetric, anti-symmetric and transitive.

Solution : We have $a \leq a, \forall a \in Z$

i.e, $(a, a) \in R, \forall a \in Z$

$\therefore$ R is reflexive.

Again, if $a \leq b$, $b \nleq a$

i.e; $(a, b) \in R \nRightarrow (b, a) \in R$

$\therefore$ R is not symmetric.

Again, $a \leq b, b \leq a \Rightarrow a = b$

i.e. $(a, b) \in R, (b, a) \in R \Rightarrow a = b$

$\therefore$ R is anti-symmetric.

Again $a \leq b, b \leq c \Rightarrow a \leq c$

i.e. $(a, b), (b, c), \in R \Rightarrow (a, c) \in R$

$\therefore$ R is transitive.

Example 9: Give an example of a relation which is transitive, but neither reflexive nor symmetric.

Solution : Let $A = \{1, 2\}$

Let $R = \{(1, 1), (1, 2)\} \subseteq A \times A$

Clearly R is a relation on A.

Here $(2, 2) \notin R$.

\ R is not reflexive.

Again $(1, 2) \in R$, but $(2, 1) \notin R$

\ R is not symmetric.

But R is transitive.

CHECK YOUR PROGRESS 2

Q 1: Let A = {1, 2, 3) and R be a reflexive relation on A having minimum number of ordered pairs. Write down R.

Q 2: Define a relation on the set

A = {1, 2, 3, 4} which is

(i) symmetric, but neither reflexive nor transitive;

(ii) reflexive and transitive, but not symmetric;

(iii) reflexive and symmetric, but not transitive.

2.5 EQUIVALENCE RELATION

Let A be a non-empty set. A relation R on A is called an equivalence relation if it is reflexive, symmetric and transitive.

Example 10: Let Z be the set of integers and

$R = \{(x, y) : x, y \in Z \text{ and } x+y \text{ is even}\}$

Examine if R is an equivalence relation on Z.

Solution : Let $x \in Z$

$\therefore$ x+x is even

$\Rightarrow (x, x) \in R, \forall\ x \in Z$

$\therefore$ R is reflexive.

$(x, y) \in R \Rightarrow x+y$ is even

$\Rightarrow y+x$ is even

$\Rightarrow (y, x) \in R$.

$\therefore$ R is symmetric.

$(x, y) \in R, (y, z) \in R$

$\Rightarrow$ x+y is even, y+z is even

$\Rightarrow$ (x+y)+(y+z) is even

$\Rightarrow$ (x+z)+2y is even

$\Rightarrow$ x+z is even

$(x, z) \in R$

$\therefore$ R is transitive

$\therefore$ R is an equivalence relation on Z.

Example 11: Let A be the set of all straight lines in a plane.

Let $R = \{(x, y) : x, y \in A \text{ and } x \perp y\}$

Examine if R is reflexive, symmetric and transitive.

Solution : A line cannot be perperdicular to itself, i.e. $x \not\perp x$

$\therefore (x, x) \notin R$

$\therefore$ R is not reflexive

If a line x is perpendicular to another line y, then y is perpendicular to x, i.e.

$x \perp y \Rightarrow y \perp x$

$\therefore (x, y) \in R \Rightarrow (y, x) \in R$

$\therefore$ R is symmetric.

If $x \perp y$, $y \perp z$, then $x \not\perp z$

$\therefore (x, y), (y, z) \in R \not\Rightarrow (x, z) \in R$

If x is perpendicular to y, y is perpendicular to z, then x is parallel to z.

$\therefore$ R is not transitive.

Example 12: Let A be the set of all straight lines in a plane.

Let $R = \{(x, y) : x, y \in A \text{ and } x||y\}$

Examine if R is an equivalence relation on A.

Solution : A line is parallel to itself, i.e. $x||x \ \forall \ x \in A$

$\therefore (x, x) \in R, \ \forall x \in R$

$\therefore$ R is reflexive.

$x||y \Rightarrow y||x$

$\therefore (x, y) \in R \Rightarrow (y, x) \in R$

$\therefore$ R is symmetric.

$x||y, y||z \Rightarrow x||z$

$\therefore (x, y), (y, z) \in R \Rightarrow (x, z) \in R.$

$\therefore$ R is transitive

Thus, R is an equivalence relation on A.

Example 13: Let IN be the set of natural numbers. Let a relation R be defined on IN x IN by (a, b) R (c, d) if and only if ad = bc

Show that R is an equivalence relation on IN x IN.

Solution : We have ab = ba

$\therefore$ (a, b) R (a, b)

$\therefore$ R is reflexive

(a, b) R (c, d) $\Rightarrow$ ad = bc; a, b, c, d, $\in$ IN

$\Rightarrow$ cb = da

$\Rightarrow$ (c, d) R (a, b)

$\therefore$ R is symmetric

(a, b) R (c, d) and (c, d) R (e, f); a, b, c, d, e, f, $\in$ IN

$\Rightarrow$ ad = bc and cf = de

$\Rightarrow$ adcf = bcde

$\Rightarrow$ af = be

$\Rightarrow$ (a, b) R (e, f)

$\therefore$ R is transitive.

Thus, R is an equivalence relation on IN x IN.

Example 14: (Congruence modulo n)

Let Z be the set of integers, and n be any fixed positive integer.

Let a, b$\in$ Z

a is said to be congruent to b modulo n if and only if a–b is divisible by n.

Symbolically, we write.

$a \equiv b \pmod{n}$

Show that the relation 'congruence modulo n' is an equivalence relation on Z.

Solution : We know that a–a is divisible by n, i.e., $a \equiv a \pmod{n}$

$\therefore$ the relation is reflexive.

Let $a \equiv b \pmod{n}$

$\Rightarrow$ a–b is divisible by n.

$\Rightarrow$ b–a is divisible by n.

$\Rightarrow b \equiv a \pmod{n}$

$\therefore$ the relation is symmetric.

Let $a \equiv b \pmod{n}$, $b \equiv c \pmod{n}$

$\Rightarrow$ a–b is divisible by n, b–c is divisible by n

$\Rightarrow$ a–b+b–c is divisible by n

$\Rightarrow$ a–c is divisible by n

$\Rightarrow a \equiv c \pmod{n}$

$\therefore$ the relation is transitive.

Thus, the relation 'congruence modulo n' is an equivalence relation on Z.

> We know that 15-3 is divisible by 4.
>
> $\therefore$15 is congruent to 3 modulo 4 i.e.
>
> $15 \equiv 3 \pmod{4}$
>
> 15–3 is not divisible by 7
>
> $\therefore$15 is not congruent to 3 modulo 7
>
> i.e. $15 \not\equiv 3 \pmod{7}$

Example 15: Let Z be the set of integers.

Let R = {(a, b) : a, b $\in$ Z, ab$\geq$0}

Examine if R is an equivalence relation on Z.

Solution : We have aa $\geq$0

\ $(a, a) \in R, \forall a \in Z$

\ R is reflexive

Let $(a, b) \in R$

\ $ab \geq 0$

$\Rightarrow ba \geq 0$

$\Rightarrow (b, a) \in R$

\ R is symmetric.

We have

$(-2) \times 0 = 0, 0 \times 2 = 0$

\ $(-2, 0), (0, 2) \in R$

But $(-2) \times 2 = -4 < 0$

\ $(-2, 2) \notin R$

\ R is not transitive.

Thus, R is not an equivalence relation.

Theorem 1: The inverse of an equivalence relation is also an equivalence relation.

Proof. Let A be a non-empty set. Let R be an equivalence relation on A

R is reflexive.

\ $(x, x) \in R, \forall x \in A$

$\Rightarrow (x, x) \in R^{-1}, \forall x \in A$

\ R^{-1} is reflexive.

Let $(x, y) \in R^{-1}$

This $\Rightarrow (y, x) \in R$ [by def^n of R^{-1}]

$\Rightarrow (x, y) \in R$ [\ R is symmetric]

$\Rightarrow (y, x) \in R^{-1}$

\ R^{-1} is symmetric.

Let $(x, y), (y, z) \in R^{-1}$

$\Rightarrow (y, x), (z, y) \in R$ [by def^n of R^{-1}]

$\Rightarrow (z, y), (y, x) \in R$

$\Rightarrow (z, x) \in R$ [\ R is transitive]

$\Rightarrow (x, z) \in R^{-1}$

\ R^{-1} is transitive

Thus, R^{-1} is an equivalence relation.

Theorem 2: The intersection of two equivalence relations is also an equivalence relation.

Proof. Let A be a non-empty set.

Let R and S be two equivalence relations on A.

R and S are reflexive

\ $(x, x) \in R$ and $(x, x) \in S$, $\forall x \in A$

$\Rightarrow (x, x) \in R \cap S, \forall x \in A$

\ $R \cap S$ is reflexive.

Let $(x, y) \in R \cap S$

This $\Rightarrow (x, y) \in R$ and $(x, y) \in S$

$\Rightarrow (y, x) \in R$ and $(y, x) \in S$ [\ R, S are symmetric]

$\Rightarrow (y, x) \in R \cap S$

\ $R \cap S$ is symmetric.

Let $(x, y) \in R \cap S$, $(y, z) \in R \cap S$

$\Rightarrow (x, y) \in R$ and $(x, y) \in S$, $(y, z) \in R$ and $(y, z) \in S$

$\Rightarrow (x, z) \in R$ and $(x, z) \in S$ [\ R, S are transitive]

$\Rightarrow (x, z) \in R \cap S$

\ $R \cap S$ is transitive.

Thus, $R \cap S$ is an equivalence relation.

Remarks : The union of two equivalence relations is not necessarily an equivalence relation.

Let us consider $A = \{1, 2, 3\}$

Now, $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$

$S = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

are two equivalence relations on A.

$R \cup S = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1), (1, 2), (2, 1)\}$

$(2, 1) \in R \cup S, (1, 3) \in R \cup S)$

But $(2, 3) \notin R \cup S$

\ $R \cup S$ is not transitive

\ $R \cup S$ is not an equivalence relation.

2.5.1 Equivalence Class

Let us consider the set Z of integers

Let R = {(a, b) : a, b $\in$ Z, a–b is divisible by 3}

i.e. R = {(a, b) : a, b$\in$Z, a $\equiv$b (mod 3)}

Clearly R is reflexive, symmetric and transitive.

\ R is an equivalence relation on Z.

Let [0] denote the set of integers congruent to 0 modulo 3. Then

[0] = {...., –9, –6, –3, 0, 3, 6, 9, ...}

Let [1] denote the set of integers congruent to 1 modulo 3. Then

[1] = {..., –8, –5, –2, 1, 4, 7, 10, ...}

Let [2] denote the set of integers congruent to 2 modulo 3. Then

[2] = {...., –7, – 4, –1, 2, 5, 8, 11, ...}

We see that

...... = [0] = [3] = [6] =.....

...... = [1] = [4] = [7] =.....

...... = [2] = [5] = [8] =.....

Each of [0], [1], [2] is called an equivalence class. [0] is the equivalence class of 0, [1] is the equivalence class of 1, [2] is the equivalence class of 2.

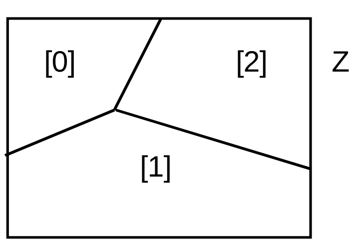
We see that there are 3 distinct equivalence classes, viz, [0], [1], [2].

Also we note that

[0] U [1] U [2] = z

[0] $\cap$ [1] = f , [1] $\cap$ [2] = f [2]$\cap$[0] = f

The set {[0], [1], [2]} is called a partition of Z.



Definition : Let A be a non-empty set and R be a relation on A.

For any a$\in$ A, the equivalence class [a] of a is defined by

If A, B are two non-empty sets, then either $A \cap B \neq \phi$ or $A \cap B = \phi$

$[a] = \{x \in A : xRa\}$

i.e. the equivalence class [a] of a is the collection of all those elements of A which are related to a under the relation R.

Note : $[a] \neq \phi$, $\because$ aRa

Definition : Let A be a non-empty set.

The set P of non-empty subsets of A is called a partition of A if

(i) A is the union of all members of P,

(ii) any two distinct members of P are disjoint.

Theorem 3 : Let A be a non-empty set and R be an equivalence relation of A. Let a, b$\in$ A. Then [a] = [b] if and only if (a, b) $\in$R.

Proof. Let [a] = [b]

$\because$ R is reflexive, $\therefore$ aRa

$\therefore$ a$\in$[a]

$\Rightarrow$ a$\in$[b]   [$\because$ [a]=[b]]

$\Rightarrow$ aRb

$\Rightarrow$ (a, b) $\in$ R

Conversely : Let (a, b) $\in$R. $\therefore$ aRb

Let x $\in$ [a]. $\therefore$ xRa

Now, xRa and aRb

$\therefore$ xRb   [$\because$ R is transitive]

$\Rightarrow$ x$\in$ [b]

Thus, x $\in$ [a] $\Rightarrow$ x$\in$[b]

$\therefore$[a] $\subseteq$ [b]   ... (1)

Again, let y$\in$[b] $\therefore$ yRb

Now, yRb and bRa   [$\because$ R is symmetric, aRb $\Rightarrow$ bRa]

$\therefore$ yRa   [$\because$ R is transitive]

$\Rightarrow$ y$\in$ [a]

Thus, y$\in$[b] $\Rightarrow$ y$\in$[a]

$\therefore$ [b] $\subseteq$ [a]   ... (2)

From (1) and (2),   [a] = [b]

Theorem 4 : Two equivalence classes are either equal or disjoint.

Proof. Let A be a non-empty set. Let R be an equivalence relation on A.

Let a, b$\in$A.

Then [a], [b] are either not disjoint or disjoint.

Let [a], [b] be not disjoint,

i.e. [a] $\cap$[b] $\neq f$.

Let x $\in$ [a] $\cap$[b]

$\Rightarrow$ x$\in$ [a] and x$\in$[b]

$\Rightarrow$ xRa and xRb

$\Rightarrow$ aRx and xRb [**Q** R is symmetric, xRa $\Rightarrow$aRx]

$\Rightarrow$ aRb [**Q** R is transitive]

Let y$\in$[a]

$\Rightarrow$ yRa

Now yRa and aRb

$\therefore$ yRb [**Q** R is transitive]

$\Rightarrow$ y $\in$ [b]

Thus, y$\in$[a] $\Rightarrow$ y$\in$[b]

$\therefore$ [a] $\subseteq$ [b]

Similarly, [b] $\subseteq$[a]

$\therefore$[a] = [b]

So, if [a], [b] are not disjoint, they are equal.

$\therefore$ [a], [b] are either equal or disjoint.

CHECK YOUR PROGRESS 3

Q 1: What is the name of the relation on a set . which is reflexive, symmetric and transitive?

Q 2: Is the relation "<'' on the set of narural nimbers IN an equivalence relation? Justify your answer.

Q 3: Let Z be the set of integers.

Let R = {(a, b) : a, b $\in$ Z, a–b is divisible by 5}

Examine if R is an equivalence relation on Z ?

Q 4: Let A be the set of all triangles in a plane.

Let R = {(x, y) : x, y $\in$ A., x is similar to y}

Examine if R is an equivalence relation on A.

Q 5: let Z be the set of integers.

Let $R = \{(x, y) : x, y \in Z, x-y \text{ is even}\}$

Is R an equivalence relation on Z ?

Justify your answer.

2.6 CONCEPT OF FUNCTION

Let A and B be two non-empty sets, and $f \subseteq A \times B$ such that

(i) $(x, y) \in f$, $\forall x \in A$ and any $y \in B$

(ii) $(x, y) \in f$ and $(x, y') \in f \Rightarrow y = y'$.

In this case f is said to be a function (or a mapping) from the set A to the set B. Symbolically we write it as $f : A \to B$.

Here A is called the domain and B is called the codomain of f.

Example 16 : Let $A = \{1, 2\}$, $B = \{7, 8, 9\}$

$f = \{(1, 8), 2, 7)\} \subseteq A \times B$.

Here, each element of A appears as the first component exactly in one of the ordered pairs of f.

$\therefore$f is a function from A to B.

Example 17 : Let $A = \{1, 2\}$, $B = \{7, 8, 9\}$

$g = \{(1, 7), (1, 9)\} \subseteq A \times B$

Here, two distinct ordered pairs have the same first component.

$\therefore$ f is not a function from A to B.

Example 18 : Let $A = \{1, 2, 3, 4\}$, $B = \{x, y, z, w\}$

Are the following relations from A to B be functions?

(i) $f_1 = \{(1, x), (1, w), (2, x), (2, z), (4, w)\}$

(ii) $f_2 = \{(1, y), (2, z), (3, x), (4, w)\}$

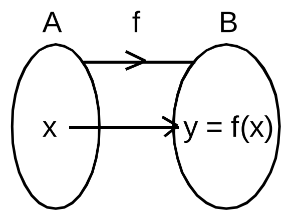
Solution : (i) No. Here two distinct ordered pairs (1, x), (1, w) have the same first component.

(ii) Yes.

Here, each element of A appears as the first component exactly in one of the ordered pairs of f_2.

Thus we see that.

Every function is a relation, but every relation is not a function.



We observe that if A and B are two non-empty sets and if each element of A is associated with a unique element of B, then the rule by which this association is made, is called a function from the set A to the set B. The rules are denoted by f, g etc. The sets A, B may be the same.

Let f be a function from A to B i.e.

$f : A \to B$. The unique element y of B that is associated with x of A is called the image of x under f. Symbolically we write it as f = f(x). x is called the pre-image of y. The set of all the images under f is called the range of f.

Example 19 : Let IN be the set of natural members, and Z be the set of integer and $f : IN \to Z$, $f(x) = (-1)^x$; $x \in IN$

Clearly, domain of f = IN

codomain of f = Z

Now $f(1) = (-1)^1 = -1$, $f(2) = (-1)^2 = 1$, $f(3) = (-1)^3 = -1$ and so on.

$\therefore$ range of f = {–1, 1}

2.6.1 Identity Function

Let A be a non-empty set and $i : A \to A$, $i(x) = x$, $\forall x \in A$

i is called the identity function.

Note : In case of identity function, domain and codomain are the same.

2.6.2 Constant Function

Let A, B be two non-empty sets and $f : A \to B$ be a function such that

$f(x) = k$, $\forall x \in A$

f is called a constant function.

Note : The range of a constant function is a singleton set.

2.7 TYPES OF FUNCTION

Let A, B be two non-empty sets and $f : A \to B$ be a function.

1. If there is at least one element in B which is not the image of any

element in A, then f is called an “into” function.

2. If each element in B is the image of at least one element in A, then f is called an “onto” function (or a surjective function or a surjective).

Note : In case of an onto function, range of f=codomain of f.

3. If different elements in A have different images in B, then f is called a one-one function (or an injective function or an injection).

4. If two (or more) different elements in A have the same image in B, then f is called a many-one function.

5. A function is said to be bijective if it is one-one (injective) and onto (surjective).

Note : Identity function is a bijective function.

How to prove that f is one-one?

Let $x_1, x_2, \in A$

If $f(x) = f(x_2) \Rightarrow x_1 = x_2$, then f is one-one.

or,

If $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, then f is one-one.

How to prove that f is onto?

Let $y \in B$ (codomain)

Let $y = f(x)$

We find x in terms of y

If $x \in A$, then f is onto; otherwise not.

Example 20: Let IN be the set of natural numbers,

Let $f : IN \rightarrow IN$, $f(x) = 3x+7$

Examine if f is a bjective function.

Solution : Let $x_1, x_2 \in IN$ (domain)

Now $f(x_1) = f(x_2)$

$\Rightarrow 3x_1+7=3x_2+7$

$\Rightarrow x_1 = x_2$

$\therefore$f is one-one (injection).

Let $y \in IN$ (codomain)

Let $y = f(x)$

$\Rightarrow y = 3x + 7$

$\Rightarrow$ x = $\frac{y-7}{3}$

If y = 1, x = $\frac{1-7}{3}$ = -2$\notin$ N

$\therefore$ f is not onto

$\therefore$ f is not a bijection.

Example 21: Let X be the set of real numbers excluding 1. Show that the function f : X$\rightarrow$X, f(x)=$\frac{x+1}{x-1}$ is one-one and onto.

Solution: Let $x_{1,}x_2 \in$ X(domain)

Let $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1+1}{x_1-1} = \frac{x_2+1}{x_2-1}$$

$$\Rightarrow \frac{(x_1+1)+(x_1-1)}{(x_1+1)-(x_1-1)} = \frac{(x_2+1)+(x_2-1)}{(x_2-1)-(x_2-1)}$$ (by componendo and dividendo)

$$\Rightarrow \frac{2x_1}{2} = \frac{2x_2}{2}$$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

Let y $\in$ X (codomain)

Let y = f(x)

$$\Rightarrow y = \frac{x+1}{x-1}$$

$\Rightarrow$ y(x–1) = x+1

$\Rightarrow$ yx–y = x+1

$\Rightarrow$ yx – x = y+1

$\Rightarrow$ x(y–1) = y+1

$\Rightarrow x = \frac{y+1}{y-1} \in$ X (domain)

$\therefore$ f is onto.

Example 22: Let A = {1, 2, 3,}. Write down all the bijective function from A to itself.

Solution :

i : A$\rightarrow$ A, i(1) = 1, i(2) = 2, i(3) = 3

f_1 : A$\rightarrow$ A, $f_1(1) = 1$, $f_1(2) = 3$, $f_1(3) = 2$

f_2 : A$\rightarrow$ A, $f_2(1) = 2$, $f_2(2) = 1$, $f_2(3) = 3$

$f_3 : A \to A$, $f_3(1) = 3$, $f_3(2) = 2$, $f_3(3) = 1$

$f_4 : A \to A$, $f_4(1) = 2$, $f_4(2) = 3$, $f_4(3) = 1$

$f_5 : A \to A$, $f_5(1) = 3$, $f_5(2) = 1$, $f_5(3) = 2$

Note. There are 6 =3! bijective functions from A = {1, 2, 3} to itself.

If A has n elements, there are n! bijective functions from A to itself.

Theorem 5 : Let A be a finite set, and $f : A \to A$ be onto. Then f is one-one.

Proof. Let A be a finite set having n elements.

Let $A = \{a_1, a_2, ..., a_n)$, where a_i' s are distinct.

Let $f : A \to A$ be onto.

Now, range of f

$= \{f(a_1), f(a_2), ..., f(a_n)\}$

$\therefore$ f is onto, range of f = codomain of f = A.

$\therefore A = \{f(a_1), f(a_2), ..., f(a_n)\}$

$\therefore$ A has n elements,

$\therefore f(a_1), f(a_2), ..., f(a_n)$ are distinct.

Thus, distinct elements in A (domain) have distinct images in A (codomain).

$\therefore$ f is one-one.

Note : The result does not hold good if A is an infinite set.

Let IN be the set of natural numbers.

Let $f : IN \to IN$, $f(x) = 1$, if x = 1 and x = 2,

$= x-1$, if $x \geq 3$

Clearly, f is onto, but not one-one.

Theorem 6 : Let A be a finite set, and $f:A \to A$ be one-one. Then f is onto.

Proof : Let A be a finite set having n elements.

Let $A = \{a_1, a_2, ..., a_n\}$, where a_i's is are distinct.

Let $f : A \to A$ be one-one.

$\therefore f(a_1), f(a_2), ..., f(a_n)$ are n distinct elements of A (codomain).

Let $u \in A$ (codomain).

Let $u = f(a_i)$, $1 \leq i \leq n$

$\therefore$ there exists $a_i \in A$ (domain) such that $f(a_i) = u$.

$\therefore$ f is onto.

Note : The result does not hold good if A is an infinite set.

Let IN be the set of natural numbers.

Let f : IN →IN, f(x) = 5x

Clearly, f is one-one, but not onto.

CHECK YOUR PROGRESS 4

Q 1: Let IN be the set of natural numbers.

Let f : IN →IN, $f(x) = x^2+1$

Examine if f is (i) one-one, (ii) onto.

Q 2: Let IR be the set of real numbers and

f : IR→IR be defined by

$$f(x) = \begin{cases} 1, & \text{if} \quad x > 0 \\ 0, & \text{if} \quad x = 0 \\ -1, & \text{if} \quad x < 0 \end{cases}$$

Examine if f is (i) one-one, (ii) onto

Q 3: Let f : IR→IR be defined by f (x) =|x|

Examine if f is (i) one-one, (ii) onto

$$\left[f(x) = |x| = \begin{cases} x, \text{if} & x \geq 0 \\ -x, \text{if} & x < 0 \end{cases} \right]$$

Q 4: If A = {1, 2}, write down all the bijective functions from A to itself.

Q 5: Write down the condition such that a constant function is onto.

Q 6: Write down the condition such that a constant function is one-one.

2.8 LET US SUM UP

- If A, B are two non-empty sets, a subset of A x B in said to be a relation from A to B.
- If A, B are two finite sets and n(A) = x, n(B) = y, the number of relations from A to B is 2^{xy}.
- If A is a non-empty set, I_A = {(a, a) : a∈A} is called the identity relation on A.

- If A, B are two non-empty and R is a relation from A to B, the inverse relation R^{-1} is defined by

  $R^{-1} = \{(b, a) : (a, b) \in R\}$, which is a relation from B to A.
- A relation R on a non-empty set A is called.

  (i) reflexive if $(a, a) \in R$, for all $a \in A$;

  (ii) symmetric if whenever $(a, b) \in R$, $(b, a) \in R$;

  (iii) anti-symmetric if $(a, b) \in R$, $(b, a) \in R \Rightarrow a = b$;

  (iv) transitive if whenever (a, b), $(b, c) \in R$, then $(a, c) \in R$.
- A relation R on a non-empty set A is called an equivalence relation if it is reflexive, symmetric and transitive.
- The inverse of an equivalence relation is also an equivalence relation.
- The intersection of two equivalence relations is also an equivalence relation.
- If R is a relation on a non-empty set A, then for any $a \in A$; the equivalence class [a] of a is the collection of all those elements of A which are related to a under the relation R.
- Two equivalence classes are either equal or disjoint.
- If A and B are two non-empty sets and if each element of A is associated with a unique element of B, then the rule by which this association is made, is called a function from A to B.
- Every function is a relation, but every relation is not a function.
- If different elements in domain have different images in codomain, then the function is one-one (injective).
- If each element in codomain is the image of at least one element in domain then the function is onto (surjective).
- A function is bijective if it is injective and surjective.
- If A is a finite set and $f : A \rightarrow A$ is onto, then f is one-one.
- If A is a finite set and $f : A \rightarrow A$ is one-one, then f is onto.

2.9 ANSWERS TO CHECK YOUR PROGRESS

CHECK YOUR PROGRESS – 1

Ans to Q No 1: 2^{12}

Ans to Q No 2: 2^{25}

Ans to Q No 3: (7, 4), (72, 27) belong to R.

Ans to Q No 4: Yes. The sum of two even integers cannot be odd.

Ans to Q No 5: $I_A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$

Ans to Q No 6: $R^{-1} = \{(7, 4), (8, 5), (7, 6), (8, 6), (9, 6)\}$

CHECK YOUR PROGRESS – 2

Ans to Q No 1: $R = \{(1, 1), (2, 2), (3, 3)\}$

CHECK YOUR PROGRESS – 3

Ans to Q No 1: Equivalence relation.

Ans to Q No 2: No. Not reflexive and symmetric.

Ans to Q No 3: Equivalence relation.

Ans to Q No 4: Equivalence relation.

Ans to Q No 5: Yes.

CHECK YOUR PROGRESS – 4

Ans to Q No 1: One-one, but not onto.

Ans to Q No 2: Neither one-one nor onto.

Ans to Q No 3: Neither one-one nor onto

Ans to Q No 4: $i : A \rightarrow A$, $i(1) = 1$, $i(2) = 2$

$f : A \rightarrow A$, $f(1) = 2$, $f(2) = 1$

Ans to Q No 5: Codomain is a singleton set.

Ans to Q No 6: Domain is a singleton set.

2.10 FURTHER READINGS

1) Lanski, Charles : *Concepts in Abstract Algebra*, (First Indian edition, 2010) American Mathematical Society.

2) Vasistha, A. R. : *Modern Algebra (Abstract algebra),* Krishna Prakashan Media (p) Ltd. Meerut-250001 (U. P).

3) Vinberg, E. B. : *A course in Algebra* (First Indian edition, 2009), American Mathematical Society).

4) Vatsa, B. S. and Vatsa, Suchi : *Modern Algebra*, New Age International (p) Ltd., New Delhi-110002.

5) Sen, M. K, Ghosh, Shamik and Mukhopadhyay, Parthasarthi : *Topics in Abstract Algebra,* Universities Press (India) Pvt. Ltd., Hyderabad, 500029.

2.11 MODEL QUESTION

Q 1: Let A = {1, 2, 3}, B = {3, 4, 5}.

How many relations are there from A to B ? Write down any four relations from A to B.

Q 2: Let A = {3, 4, 5}, B = {5, 6, 7}

Which of the following relations are functions from A to B ? If it is a function, determine whether it is one-one and whether it is onto ?

(i) {(3, 5), (4, 5), (5, 7)}

(ii) {(3, 7), (4, 5), (5, 6)}

(iii) {(3, 6), (4, 6), (5, 6)}

(iv) {(3, 6), (4, 7), (5, 5)}

Q 3: Let Q be the set of rational numbers and $f : Q \rightarrow Q$ be a function defined $f(x) = 4x+5$. Examine if f is bijective

Q 4: Let A be the set of all triangles in a plane.

Let R = {(x, y) : x, y $\in$ A and x is congruent to y}

Examine if R is an equivalence relation on A.

Q 5: Let IN be the set of natural numbers. A relation R is defined on IN x IN by (a, b), R (c, d) if and only if a+d = b+c.

Show that R is an equivalence relation on IN x IN.

Q 6: Let $A = \{1, 2, 3\}$

$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$

Show that R is reflexive but neither symmetric nor transitive.

Q 7: Let C be the set of complex numbers and IR be the set of real numbers.

Let $f : C \to IR$, $f(z) = |z|$, $z \in C$.

Examine if f is (1) one-one, (ii) onto.

Q 8: Let A, B be two non-empty sets.

Let $f : A \times B \to B \times A$, $f(a, b) = (b, a)$; $(a, b) \in A \times B$

Show that f is bijective.

Q 9: Let $f : IN \times IN$, $f(a, b) = 3^a 4^b$, $(a, b) \in IN \times IN$

Examine if f is (i) one-one, (ii) onto.

Q 10: Let $f : Z \times Z \to Z$, $f(a, b) = ab$, $(a, b) \in Z \times Z$

Examine if f is (i) injective, (ii) surjective.

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UNIT 3: MATHEMATICS OF FINANCE

UNIT STRUCTURE

3.1 LEARNING OBJECTIVES

After going through this unit you will be able to:

- explain simple interest, amount and principal value.
- show simple interest and amount at the end of a given period of time.
- explain compound interest, amount and principal value.
- show compound interest and amount at the end of a given period of time when the interest is compounded yearly or half-yearly or quarterly.

- discuss about depreciation and scrap value of assets.
- explain annuity, types of annuity, to find amounts of ordinary annuity and annuity due.
- show present value of ordinary annuity and present value of annuity due.

3.2 INTRODUCTION

A man takes loan from another man or financial institution, whenever he is in need of money. The man (or the institute) who lends money charges extra money to be paid by the debtor. Therefore the debtor (who takes the loan) is to pay some extra money in addition to the money borrowed to clear his debt. The extra money to be paid for the loan is called interest. The borrowed money is called the principal and the time for which the loan is taken is called the period of loan or simply the period. The principal together with the interest on it is called the amount.

For calculation of interest on the principal, how much interest on Rs.100/= for one year is to be paid is fixed. This is called the rate of interest. It is always expressed as percent per annum. If the interest of Rs.8/= is to be paid on Rs.100/= for one year, then the rate of interest is 8% p. a. (p. a. means per annum). Thus if a man borrows Rs.500/= for 3 years at interest 8.25% p. a., then principal is equal to Rs.500/=, period is equal to 3 years and interest on Rs.100/= for one year is equal to Rs.8.25/=.

3.3 SIMPLE INTEREST

If the interest is charged only on the borrowed money (principal) throughout the period of loan, then the interest is called simple interest. Thus the simple interest is same for every year for the period of the loan.

3.3.1 Formulae for Calculation of Simple Interest, Amount and Principal

Formula for simple interest: Let Rs.P denote the principal, n denote the period of the loan (in years). Let the rate of interest be r% p. a.

Now, interest on Rs.100/= for 1 year = Rs.r

Hence, interest on Rs.1/= for 1 year = Rs. $\frac{r}{100}$

$\Rightarrow$ Interest on Rs.P for 1 year = Rs. $\frac{P \times r}{100}$

$\Rightarrow$ Interest on Rs.P for n year = Rs. $\frac{P \times n \times r}{100}$

If *I* (in rupees) denotes the interest, then

$$I = \frac{P \times n \times r}{100} \quad \text{............ (3.1)}$$

Thus if P, r, n are known, *I* can be found out using the formula (4.1).

Formula for amount: If A (in rupees) denotes the amount, then

A = P + 1 .... (3.2)

Also,

$$A = P + I$$

$$\Rightarrow A = P + \frac{P \times n \times r}{100}$$

$$\Rightarrow A = P\left(1 + \frac{r \times n}{100}\right) \mathbf{L} \; \mathbf{L} \; (3.3)$$

Thus using (4.3), A can be found out directly if P, r, n are known.

Formulae for principal: We know that

$$I = \frac{P \times r \times n}{100}$$

This implies, $P \times r \times n = I \times 100 \Rightarrow P = \frac{I \times 100}{r \times n} \mathbf{L} \; \mathbf{L} \; (4.4)$

Also, we have $P\left(1 + \frac{r \times n}{100}\right) = A \Rightarrow P = \frac{A}{1 + \frac{r \times n}{100}} \mathbf{L} \; \mathbf{L} \; (4.5)$

Thus, if *I, r, n* are known, we use (4.4) and if *A, r, n* are known, we use (4.5) to find P.

LET US KNOW

1) Unless, mentioned otherwise, interest means simple interest.
2) The period of loan must be expressed in years if it is in months or days. In this respect we have
   1 month = 1/12 year
   1 day = 1/365 year, etc.
3) When the period of loan has two terminal dates (that is, the date on which the loan is taken and the date on which the loan is repaid), then to calculate the number of days, one of the two terminal dates is excluded. For example, if a loan is taken on April 3, 2008 and it is repaid on July 17, 2008, we calculate the number of days for which interest is to be paid, as follows:

Month	**No. of days.**
April	27 (30 - 3)
May	31
June	30
July	17
Total number of days:	105

Hence, the period of the loan in years (n) = $\frac{105}{361} = \frac{21}{73}$

Note: From April 3, 2008 to July 17, 2008, number of days is equal to 106. Excluding the first terminal date April 3, we got 105 days as shown above.

Example 1: Find the interest and amount due on Rs.800/= for 5 years at 9.5% interest per annum.

Solution: Here P = Rs.800/-, n = 5 years, r = 9.5

Hence, interest I = Rs. $\frac{800 \times 9.5 \times 5}{100}$ = Rs. 380/-

Also, A = Rs.800/= + Rs.380/-

= Rs.1180/-.

Example 2: A man borrows Rs.12,000/= for 2 years 3 months at 12.5% simple interest per annum. How much money he is to pay to clear his debt?

Solution: Given, P = Rs.12,000/=, r = 12.5, n = 2 + $\frac{3}{12}=\frac{9}{4}$ years.

Using $A=P\left(1+\frac{r\times n}{100}\right)$, we get (in rupees)

$$A=12{,}000\times\left(1+\frac{12.5\times 9}{100\times 4}\right)=12{,}000\times\left(\frac{400+112.5}{400}\right)$$

$$=12{,}000\times\frac{512.5}{400}=15{,}375$$

Hence the required amount to be paid is equal to Rs.15,375/-.

Example 3: A man lent one part of Rs.15,000/= to a man at 5% p.a. and the other part at 6% p.a. interest to another man and he received Rs.17,460/= as total amount after 3 years. What sum was lent to each man?

Solution: Let the sum of money lent at 5% p.a. interest be Rs.x. Therefore the sum of money lent at 6% p.a. is Rs.(15,000 – x).

Also, in 3 years total interest received = Rs.(17,460 – 15,000)

= Rs.2460/=

Now, interest on Rs.x = Rs. $\frac{x\times 5\times 3}{100}$ = Rs. $\frac{15x}{100}$ and interest on Rs.(15,000 – x) = Rs. $\frac{(15000-x)\times 6\times 3}{100}=Rs.\frac{18\times(15{,}000-x)}{100}$.

Hence, $\frac{15x}{100}+\frac{18\times(15000-x)}{100}=2460$

$$\Rightarrow 15x+270000-18x=246000$$
$$\Rightarrow -3x=-24000$$
$$\Rightarrow x=8000$$

This implies that the money lent to the first man = Rs.8,000/= and the money lent to the second man = Rs.(15,000 – 8,000) = Rs.7,000/=.

Exercise 4.4: What sum of money will amount to Rs.34,125/= at 6.25% interest per annum in 3 years and 6 months?

Solution: Given, A = Rs.34,125/=, r = 6.25, n = $3\frac{6}{12}=\frac{7}{2}$ years.

Now, $P = \dfrac{A}{1+\dfrac{r\times n}{100}} = Rs.\left(\dfrac{34{,}125}{1+\dfrac{6.25\times 7}{100\times 2}}\right) = Rs.\left(\dfrac{34{,}125}{\dfrac{200+43.75}{200}}\right)$

$$= Rs\left(\frac{34125\times 200}{243.75}\right) = Rs.\left(\frac{3412500\times 200}{24375}\right) = Rs.(140\times 200)$$

$$= Rs.28{,}000/=$$

Hence the principal amount of the loan is Rs.28,000/=.

Exercise 4.5: A sum of money amounts to Rs.1092/= in 5 years and Rs.1243.20 in 8 years. Find the sum.

Solution: Here, Rs.1243.20 = P + interest for 8 years

Rs.1092 = P + interest for 5 years.

On subtraction, we get

Rs.(1243.20 – 1092) = interest for 3 years.

Hence, interest for 3 years = Rs.151.20

$\Rightarrow$ Interest for 1 year = Rs. $\left(\dfrac{151.20}{3}\right)$ = Rs. 50.40

$\Rightarrow$ Interest for 5 years = Rs. 252/=

Hence the principal = Rs.(1092 – 252) = Rs. 840/=.

3.4 FORMULAE FOR CALCULATION OF INTEREST AND PERIOD

Formula to find rate of interest: We know that $I = \dfrac{P\times r\times n}{100}$. This implies that $P\times r\times n = 100\times I$.

Hence, $r = \dfrac{100\times I}{P\times n}$ …… (3.6)

Also, $I = A - P$.

Thus, if *P, I, n* or *A, P, n* are known, we can find the rate of interest.

Exercise 4.6: At what rate of interest a sum of money becomes triple itself in 20 years?

Solution: Let the principal be P. Given that n = 20 years and A = 3P.

This implies that *I* = 3*P* – *P* = 2*P*.

Now, from the formula (4.6) we find that

$$r = \frac{I \times 100}{P \times n} = \frac{2P \times 100}{P \times 20} = 10.$$

Hence the rate of interest is equal to 10% per annum.

Formula to find period: We know that $I = \frac{P \times r \times n}{100}$. This implies that

$P \times r \times n = 100 \times I$.

Hence, $n = \frac{100 \times I}{P \times r}$ $\quad \ldots\ldots (3.7)$

Also, $I = A - P$.

Thus, if *P, I, r* or *A, P, r* are known, we can find the period of the loan.

Exercise 4.7: In what time a sum of Rs.12,600/= will amount to Rs.15,976.80 at 8.04% interest per annum?

Solution: Given that P = Rs.12,600/=

A = Rs.15,976.80

r = 8.04

Also, I = Rs.(15,976.80 – 12,600) = Rs.3,376.80

From the formula (4.7) we get,

$n = \frac{100 \times I}{P \times r} = \frac{3376.80 \times 100}{12600 \times 8.04} = \frac{10}{3} = 3\frac{1}{3}$ years.

This implies that period is equal to 3 years and 4 months.

CHECK YOUR PROGRESS

Q 1: State whether the following statements are True (T) or False (F):

(i) The interest is counted on the principal.(T / F)

(ii) If *i* is the interest on Rs.1/= for 1 year, then $I = P \times i \times n$, where *I, P, n* have the usual meaning. (T / F)

(iii) The interest on Rs.450/= for 1 month at 12% p.a. simple interest is Rs.5.40. (T / F)

(iv) A man borrowed Rs.1000/= on July 13, 2008 and repaid along with the interest on it on December 6, 2008. Then the number of days for which interest was counted is 147. (T / F)

(v) The interest on a given sum of money becomes one forth of it in 2 years if the rate of interest is 12.5% per annum. (T / F)

Q 2: Fill up the blanks:

(i) The interest on Rs.1250 for 2 years 6 months at the rate of interest 6.25% p.a. is ………………………

(ii) A man borrows Rs.750 for 2 years 1 month at 12% interest per annum. The amount he is required to pay to clear his debt is ……….

(iii) The rate of interest for which the interest of a certain sum of money will be $2/5$ th of the sum in 5 years is.....................

(iv) A sum of money amounts to Rs.670 in 4 years and the same sum amount to Rs.755 in 6 years. The sum of money and the rate of interest are........................ and .... respectively.

3.5 COMPOUND INTEREST

We have already learnt how to calculate simple interest on the principal for the period of loan at a given rate of interest per annum. The simple interest on the principal is same for every year. The financial institutions like Banks, Life Insurance Corporation etc charge interest other than the simple interest. This type of interest is known as compound interest. In this case, interest on the principal for first year (or first six months or first 3 months) of the period of the loan is added to the principal for the second year (or second six months or second 3 months) of the period. This process is continued for the period of the loan. The difference between the original principal and the amount at the end of the period of loan is called the compound interest. It is to be noted that

(1) If interest is compounded half yearly (that is, six monthly), then interest on the principal for the first six months is added to the principal and the sum is taken as principal for the next six months and the process is continued for the period of the loan.

(2) Similarly, if interest is compounded quarterly (that is, after every 3 months), then interest on the principal for the first 3 months is added

to the principal and the sum is taken as principal for the next 3 months and the process is continued for the period of the loan.

3.5.1 Formulae for Calculation of Compound Interest (C. I.), Amount and Principal

Yearly calculation:

Let rate of compound interest be *r* % per annum.

$\therefore$ Interest on Re 1 for 1 year = $Rs.\frac{r}{100}$ If the principal is *Rs. P,* then interest on *Rs. P* for 1 year is equal to $Rs.\frac{P\times r}{100}$. The amount at the end of the first year = $Rs.\left(P+\frac{P\times r}{100}\right)=Rs.P\times\left(1+\frac{r}{100}\right) \ldots\ldots (3.8)$

Hence, the principal for the 2nd year = $Rs.P\times\left(1+\frac{r}{100}\right)$. The interest on $Rs.P\times\left(1+\frac{r}{100}\right)$ for the 2nd year = $Rs.\left\{P\times\left(1+\frac{r}{100}\right)\times\frac{r}{100}\right\}$

Hence, amount at the end of 2nd year

$$= Rs.P\times\left(1+\frac{r}{100}\right)+Rs.\left\{P\times\left(1+\frac{r}{100}\right)\times\frac{r}{100}\right\}$$

$$= Rs.\left\{P\times\left(1+\frac{r}{100}\right)\times\left(1+\frac{r}{100}\right)\right\}$$

$$= Rs.P\times\left(1+\frac{r}{100}\right)^2 \ldots\ldots (3.9)$$

Similarly amount at the end of 3rd year = $Rs.P\times\left(1+\frac{r}{100}\right)^3 \ldots\ldots (3.10)$

Thus we conclude that if Rs. *A* is the amount at the end of *n*th year, then we get,

$$A=P\times\left(1+\frac{r}{100}\right)^n \ldots\ldots (3.11)$$

$$OR \quad A=P\times(1+i)^n \quad \ldots\ldots (3.12)$$

where *i* is the interest on Re 1 for 1 year = $Rs.\,{}^{r}\!/_{100}$

Again, compound interest C. I. = A – P

$$= P \times \left(1 + \frac{r}{100}\right)^n - P$$

Hence,

$$C.I. = P \times \left\{\left(1 + \frac{r}{100}\right)^n - 1\right\} \quad \mathbf{L\ L}\ (3.13)$$

$$OR \quad C.I. = P \times \left\{(1+i)^n - 1\right\} \quad \mathbf{L\ L}\ (3.14)$$

where i is the interest on Re 1 for 1 year = $Rs.\ {}^{r}\!/\!{}_{100}$

Half-yearly calculation: Interest on Re 1 for six months = $Rs.\left(\frac{r/2}{100}\right)$

and number of six months in n years is equal to $2n$. From formula (4.11) we get

$$A = P \times \left(1 + \frac{r/2}{100}\right)^{2n} \quad \mathbf{L\ L}\ (3.15)$$

$$OR \quad A = P \times \left(1 + \frac{i}{2}\right)^{2n} \quad \mathbf{L\ L}\ (3.16)$$

where $i = Rs.\ {}^{r}\!/\!{}_{100}$

Also,

$$C.I. = P \times \left\{\left(1 + \frac{r/2}{100}\right)^{2n} - 1\right\} \quad \mathbf{L\ L}\ (3.17)$$

$$OR \quad C.I. = P \times \left\{\left(1 + \frac{i}{2}\right)^{2n} - 1\right\} \quad \mathbf{L\ L}\ (3.18)$$

where $i = Rs.\ {}^{r}\!/\!{}_{100}$

Quarterly calculation: Interest on Re 1 for three months = $Rs.\left(\frac{r/4}{100}\right)$ and number of 3 months in n years is equal to $4n$. From formula (4.11) we get

$$A = P \times \left(1 + \frac{r/4}{100}\right)^{4n} \quad \mathbf{L\ L}\ (3.19)$$

$$OR \quad A = P \times \left(1 + \frac{i}{4}\right)^{4n} \quad \mathbf{L\ L}\ (3.20)$$

where $i = Rs.\, r/100$

Also,

$$C.I. = P \times \left\{\left(1 + \frac{r/4}{100}\right)^{4n} - 1\right\} \quad \mathbf{L\ L}\ (3.21)$$

$$OR \quad C.I. = P \times \left\{\left(1 + \frac{i}{4}\right)^{4n} - 1\right\} \quad \mathbf{L\ L}\ (3.22)$$

where $i = Rs.\, r/100$

Exercise 4.8: Find the amount and compound interest on Rs.5000 for 3 years at 10% p.a.

Solution: Here P = Rs.5000, n = 3 years, r = 10, $i = \frac{10}{100} = 0.1$

Now,

$$A = P \times (1+i)^n = 5000 \times (1+0.1)^3$$

$$= 5000 \times (1.1)^3 = 5000 \times 1.331$$

$$= 6655$$

Hence the required amount is Rs.6,655/- and the compound interest is equal to Rs.6655 – Rs.5000 = Rs.1,655/-.

Exercise 4.9: Find the compound interest on Rs.25,000/- at 6% p.a. for 1 year 6 months, interest being compounded half-yearly.

Solution: Here P = Rs.25,000, $n = 1\frac{1}{2}$ years = $\frac{3}{2}$ years

No. of half-years = $2n = 2 \times \frac{3}{2} = 3$, $i = \frac{6}{100} = 0.06$

Now,

$$A = P\times\left(1+\frac{i}{2}\right)^{2n} = 25000\times\left(1+\frac{0.06}{2}\right)^{3}$$

$$= 25000\times(1.03)^{3} = Rs.27318.28 \quad (approx))$$

which is the required amount. Hence the compounded interest is equal to Rs.27318.28 – Rs.25000 = Rs.2,318.28.

Exercise 4.10: A man deposited Rs.20,000/- in a Bank at 8% p.a. compound interest. If the interest is compounded quarterly, what sum of money he will get from the Bank at the end of one year?

Solution: Given P = Rs.20,000, n = 1 year, $i = \frac{8}{100} = 0.08$

No. of quarters = $4n = 4\times1 = 4$

The sum of money the person will get from the Bank is equal to

$$A = P\times\left(1+\frac{i}{4}\right)^{4n} = 20000\times\left(1+\frac{0.08}{4}\right)^{4}$$

$$= 20000\times(1.02)^{4} = 20000\times1.0824 = Rs.21,648 \quad (approx)$$

Exercise 4.11: The difference between the compound interest and simple interest on a sum of money for 3 years at 6% p.a. is Rs.545.80. Find the sum of money.

Solution: Let the sum of money be Rs.P.

Now, $i = \frac{6}{100} = 0.06, \quad n = 3$. Hence,

A (at compound interest) = $P\times(1+i)^{n} = P\times(1.06)^{3} = P\times1.190916$

$$\therefore C.\,I. = P\times1.190916 - P = P\times0.190916$$

Also, simple interest (*S. I.*) = $\frac{P\times6\times3}{100} = P\times0.18$

It is given that *C. I. – S. I.* = 545.80. This implies that

$$P\times0.190916 - P\times0.18 = 545.80$$

$$\Rightarrow P\times0.01096 = 545.80$$

$$\Rightarrow P = \frac{545.80}{0.010916} = 50000$$

Hence the required sum of money = Rs.50,000/-.

Exercise 4.12: A sum of money amounts to Rs.26,620 in 3 years and amounts to Rs.32,210.20 in 5 years at compound interest. Find the sum of money and the rate of interest.

Solution: Let the sum of money be Rs. P and the rate of compound interest be *r* % p.a. It is given that

$$P \times (1+i)^3 = 26{,}620 \qquad \mathbf{L}\ (1)$$
$$P \times (1+i)^5 = 32{,}210.20 \qquad \mathbf{L}\ (2)$$

Dividing (2) by (1), we get

$$(1+i)^2 = \frac{32210.20}{26620} = 1.21$$
$$\Rightarrow \quad 1+i = 1.1 \Rightarrow \quad i = 0.1 \Rightarrow \frac{r}{100} = 0.1 \quad \Rightarrow r = 10$$

Hence the rate of interest = 10% p.a.

Putting *i* = 0.1 in (1), we get

$$P = \frac{26620}{(1.1)^3} = \frac{26620}{1.331} = 20000$$

Hence the required sum of money = Rs.20,000/-.

Exercise 4.13: At what rate percent of compound interest a given sum of money will be double itself in 15 years?

Solution: Let the principal be Rs. P and the rate of compound interest be *r* % p.a. It is given that $A = Rs.2P, \quad n = 15.$

Now,

$$P \times (1+i)^{15} = 2P \Rightarrow (1+i)^{15} = 2 \Rightarrow 15\log(1+i) = \log 2$$
$$\Rightarrow \log(1+i) \quad = \frac{0.3010}{15} = 0.0201\,(approx)$$
$$\Rightarrow 1+i \qquad = anti\log(0.0201) = 1.047$$
$$\Rightarrow \quad i \qquad = 0.047\ (approx)$$

Hence required rate of interest =   (approx).

3.5.2 Present Value

If a sum of money is invested for a period of time at compound interest, then the amount to be obtained is called the future value and the sum of money invested is called the present value.

Thus if the sum of money invested at *r*% p.a. compounded interest is *Rs. P* and *Rs. A* is its future value, then we get

$$A = P \times (1+i)^n, \text{ where } i = Rs.\frac{r}{100}$$

$$\therefore \quad P = \frac{A}{(1+i)^n} = A \times (1+i)^{-n}$$

If interest is compounded half yearly, then $P = A \times \left(1+\frac{i}{2}\right)^{-2n}$

If interest is compounded quarterly, then $P = A \times \left(1+\frac{i}{4}\right)^{-4n}$

Exercise 4.14: Find the principal value to be invested at p.a. so that after 5 years the amount will be Rs.12,000 if the interest is compounded half yearly.

Solution: Here $A = Rs. 12{,}000, \quad i = \frac{8}{100} = 0.08, \quad n = 5$. Let the present value be *Rs. P*. Since the interest is compounded half-yearly, therefore

$$P = A \times \left(1+\frac{i}{2}\right)^{-2n} = 12000 \times (1+0.04)^{-10} = 12000 \times (1.04)^{-10}$$

$$\therefore \quad \log P = \log 12000 - 10\log 1.04 = 4.0792 - 10 \times 0.0170$$

$$= 4.0792 - 0.170 = 3.9092$$

$$\Rightarrow \quad P = anti\log 3.9092 = 8114$$

Hence the present value = Rs.8114/-.

3.6 DEPRECIATIONS

The value of asset like buildings, machines etc go on decreasing year after year since their life time goes on diminishing. The value of the depreciable asset at the end of its useful life time is called the scrap value. The total depreciation of the asset is equal to the difference between the original value and the scrap value.

If rate of depreciation is *r*% p.a. on the initial value of *Rs. P,* then the depreciated value A in Rupees at the end of years is given by

$$A = P \times \left(1 - \frac{r}{100}\right)^n \quad OR \quad A = P \times (1-i)^n, \; where \; i = Rs.\frac{r}{100}$$

Exercise 4.15: A machine costing Rs.1,00,000/- depreciates at 10% p.a. If the useful life of the machine is 10 years, find the scrap value.

Solution: Here $P = Rs.100000, \quad n = 10, \quad i = \frac{10}{100} = 0.1$

Now,

$$A = P \times (1-i)^n = 100000 \times (1-0.1)^{10} = 100000 \times (0.9)^{10}$$

$$\Rightarrow \quad \log A = \log 100000 + 10 \times \log(0.9) = 5 + 10 \times \bar{1}.9452$$

$$= 5 - 10 \times (-1 + 0.9452) = 5 - 0.548 = 4.452$$

$$\Rightarrow A = anti\log 4.452 = 28310$$

Hence the scrap value = Rs.28,310/-

Exercise 4.16: The scrap value of a machine costing Rs.10000 at the end of 10 years is equal to Rs.2785. Find the rate of depreciation.

Solution: Here *P* = *Rs.* 10000, *n* = 10, *A* = *Rs. 2785.* Let the rate of depreciation be *r*% p.a.

Now,

$$A = P \times \left(1 - \frac{r}{100}\right)^{10} \Rightarrow 2785 = 10000 \times \left(1 - \frac{r}{100}\right)^{10}$$

$$\Rightarrow \quad \left(1 - \frac{r}{100}\right)^{10} = \frac{2785}{10000} = 0.2785$$

$$\Rightarrow \quad 10 \times \log\left(1 - \frac{r}{100}\right) = \log 0.2785 = -0.55517$$

$$\Rightarrow \quad \log\left(1 - \frac{r}{100}\right) = -0.055517 = \bar{1}.944483 = \bar{1}.9445$$

$$\Rightarrow \quad 1 - \frac{r}{100} = anti\log(\bar{1}.9445) = 0.88 \Rightarrow r = 12$$

Hence the required rate of depreciation = 12% per annum.

CHECK YOUR PROGRESS

Q 3: State whether the following statements are True (T) or False (F):

(i) The simple interest and the compound interest on Rs. P at *r*% p.a. for 1 year are equal. (T / F)

(ii) The difference between the compound interest and the simple interest on Rs.1000 for 2 years at 10% p.a. interest is Rs.10. (T / F)

(iii) The sum of Rs.100 at 20% p.a. interest will amount to Rs.110.05 at the end of six months if interest is compounded quarterly. (T / F)

(iv) If the interest is compounded half yearly, the compound interest on Rs.1000 for 1 year 6 months at 10% p.a. is Rs.331. (T / F)

Q 4: Fill up the blanks:

(i) The compound interest on Rs.20000 for 3 years at 4% p.a. compound interest is equal to ....

(ii) If the interest is compounded half-yearly, the amount on Rs.25000 for 2 years 6 months at 10% p.a. compound interest is equal to ....

3.7 ANNUITY

An annuity is a fixed sum of money paid periodically under some conditions. The period of payment may be one year, six months, three months, one month, etc. Interest (simple or compound) is to be paid on each payment at the end of each period of payment. The sum of all the payments together with the interest on them is called the amount of annuity. For example, let us suppose that a man deposits Rs.500 at the beginning of each quarter in a Bank for 3 years which pays 8% interest compounded quarterly. Then size of each payment is equal to Rs.500/-, period of payment is 3 months, term of payment is 3 years, and the quarterly rate of compound interest is equal to $8/4 = 2\%$. The amount which will be paid by the Bank will be the amount of annuity.

3.7.1 Types of Annuity

(1) **Annuity certain or Ordinary annuity:** If the payment is made annually, then the annuity is called annuity certain or ordinary annuity.

(2) **Annuity due:** If the payments are made at the beginning of each payment period, the annuity is called annuity due.

(3) **Deferred Annuity:** If the annuity is such that it begins after a lapse of certain number of years, is called Deferred annuity. If the annuity is deferred for 5 years, then the first payment of the annuity is to be paid after 5 years, that is, at the end of (5+1) = 6 years.

(4) **Life Annuity:** The life annuity is an annuity which is payable for the life of a person.

(5) **Perpetual Annuity:** An annuity which is payable for ever is called perpetual annuity or perpetuity.

Remark: If not stated otherwise specifically, the payment is taken to be made annually.

3.7.2 Amounts of Ordinary Annuity and Annuity Due

(A) Formula for finding amount of Ordinary Annuity: At the end of first year, the sum of *Rs. P* is due and the amount of this sum for the remaining *(n – 1)* years at compound interest is equal to $P\times(1+i)^{n-1}$, where *i* is the interest on Rupee 1 for one year. At the end of 2nd year, another sum of *Rs. P* is due and the amount on this sum for the remaining *(n – 2)* years is equal to $P\times(1+i)^{n-2}$ and so on. If these amounts are denoted by $A_1, A_2, \ldots, A_n$ respectively, then

$$A_1 = P\times(1+i)^{n-1}, \quad A_2 = P\times(1+i)^{n-2}, \quad \ldots,$$
$$A_{n-1} = P\times(1+i)^{n-(n-1)} = P\times(1+i), \quad A_n = P\times(1+i)^{n-n} = P.$$

Now *A* is the sum of all these amounts and hence

$$A = A_1 + A_2 + \ldots + A_n = P\times(1+i)^{n-1} + P\times(1+i)^{n-2} + \ldots + P\times(1+i) + P$$
$$= P\times[1+(1+i)+(1+i)^2 + \ldots + (1+i)^{n-1}]$$

We see that the series in the bracket is a geometric series with first term a = 1 and common ratio r = 1 + i.

$$\therefore \quad A = P\times\left(\frac{a\times(r^n-1)}{r-1}\right) = P\times\left(\frac{(1+i)^n-1}{1+i-1}\right) = \frac{P}{i}\left[(1+i)^n-1\right] \quad \mathbf{L}\ (3.23)$$

(B) Formula for finding amount of Annuity Due: In case of Annuity Due, payments are made at the beginning of each period of payment. Let P be annuity, n be the number of years, rate of compound interest be r% per annum. Let $A_1, A_2, \mathbf{L}, A_n$ be the amounts of corresponding payments and be the amount of annuity. Now the payment paid at the beginning of first year earns compound interest (C. I.) for n years. Hence $A_1 = P\times(1+i)^n$, where i is the interest on Re 1 for 1 year. Similarly, $A_2 = P\times(1+i)^{n-1}$, $A_{n-1} = P\times(1+i)^2$, $A_n = P\times(1+i)$

Hence

$$A = A_1 + A_2 + \mathbf{L} + A_n$$

$$= P\times(1+i)^n + P\times(1+i)^{n-1} + \mathbf{L} + P\times(1+i)^2 + P\times(1+i)$$

$$= P\times(1+i)\times[1+(1+i)+(1+i)^2+\mathbf{L}+(1+i)^{n-1}]$$

$$= P\times(1+i)\times\frac{(1+i)^n-1}{(1+i)-1} = \frac{P\times(1+i)}{i}\times\left[(1+i)^n-1\right] \quad \mathbf{L}\ (3.24)$$

Exercise 4.17: A man deposited Rs.1500 in a Bank at the end of each year for 10 years at 5% p.a. compounded interest. What amount will be deposited to his account ?

Solution: Here $P = 1500$, $n = 10$, $i = \frac{5}{100} = 0.05$ Since the man deposited the money at the end of each year, the type of annuity is ordinary.

Now,

$$A = \frac{P\times\left[(1+i)^n-1\right]}{i} = \frac{1500\times\left[(1+0.05)^{10}-1\right]}{0.05} = 30000\times\left[(1.05)^{10}-1\right] \quad \mathbf{L}\ (1)$$

Let $x = (1.05)^{10}$. Then, $\log x = 10\times\log(1.05) = 10\times 0.02119 = 0.2119$.

This implies that $x = anti\log(0.2119) = 1.629$.

From (1), we get $A = 30000\times(1.629-1) = 18870$

Hence the required amount of annuity = Rs.18,870/-, that is, at the end of 10 years a sum of Rs.18,870/- will be deposited to his account.

Exercise 4.18: A man deposited Rs.400 at the beginning of each year in a Bank which pays 8% compounded interest per annum. Find the balance in his account at the end of 12 years.

Solution: Here $P = 400, \quad n = 12, \quad i = 0.08$. This problem is a problem of annuity due. Therefore the amount of annuity *A* is given by

$$A = \frac{P \times (1+i)}{i}\left[(1+i)^n - 1\right] = \frac{400 \times (1+0.08)}{0.08}\left[(1.08)^{12} - 1\right]$$
$$= 5400 \times \left[(1.08)^{12} - 1\right] \quad \ldots (1)$$

Let $x = (1.08)^{12}$. Then, $\log x = 12 \times \log(1.08) = 12 \times 0.033424 = 0.40108$

This implies that $x = anti\log(0.4011) = 2.519 \quad (approx)$.

From (1), we get $A = 5400 \times (2.519 - 1) = 8202.60$. Hence the balance in his account at the end of 12 years = Rs.8202.60.

CHECK YOUR PROGRESS

Q 5: State whether the following statements are True (T) or False (F):

(i) An annuity is a fixed sum of money payable at the end or at the beginning of each period of payment. (T / F)

(ii) There is no difference between ordinary annuity and annuity due. (T / F)

(iii) In case of deferred annuity, the equal installments are not paid from the beginning of the annuity. (T / F)

(iv) If the first installment of an annuity is paid at the end of 6th year and the other installments are paid regularly, then the annuity is deferred for 6 years. (T / F)

(v) In case of an annuity for *n* years, the annuity *P* is paid at the end of every six months. If *i* denotes the interest on Re 1 for six months, then the amount *A* is given by

$$A = \frac{P}{i}\left[(1+i)^{2n} - 1\right].$$ (T / F)

Q 6: Fill up the blanks.

(i) The amount of annuity of Rs100 paid at the end of every year for 10 years at 10% p.a. compound interest is -----------------

(ii) The amount of annuity of Rs.600 paid at the beginning of each year for 20 years at 12% compound interest per annum is --------------

3.7.3 Present values of Ordinary Annuity and Annuity Due

The sum of the present values of all the payments (installments) of an annuity is called the present value of that annuity.

(A) Present values of Ordinary Annuity: Let *P* (in Rupees) be the annuity, *i* the interest on Re 1 for 1 year, *n* the number of years and *V* the present value of the ordinary annuity. At the end of first year, *P* is due and the present value V_1 of *P* is given by

$P = V_1 \times (1+i),$ that is, $V_1 = P \times (1+i)^{-1} = \dfrac{P}{1+i}$. Thus, if V_2... denote the present values of the 2nd installment, …, then

$$V_2 = P \times (1+i)^{-2} = \frac{P}{(1+i)^2}, \quad \ldots, V_n = P \times (1+i)^{-n} = \frac{P}{(1+i)^n}$$

Now,

$$V = V_1 + V_2 + \ldots + V_n = \frac{P}{1+i} + \frac{P}{(1+i)^2} + \ldots + \frac{P}{(1+i)^n}$$

$$= \frac{P}{(1+i)^n} \times \left[1 + (1+i) + \ldots + (1+i)^{n-1}\right] = \frac{P}{(1+i)^n} \times \left[\frac{(1+i)^n - 1}{1+i-1}\right]$$

$$= \frac{P}{i} \times \left[1 - (1+i)^{-n}\right] \quad \ldots (3.25)$$

(B) Present value of Annuity Due: Let *P* (in Rupees) be the annuity, *i* the interest on Re 1 for 1 year and *n* the number of years. Now the amount *P* is paid at the beginning of the first year. Hence its present value V_1 is equal to *P*. The second installment amounting is paid at the beginning of the 2nd year, that is, at the end of the 1st year. If V_2 is its present value, then $P = V_2 \times (1+i) \;\Rightarrow\; V_2 = \dfrac{P}{1+i}$ Similarly, if

$V_3, \mathbf{L}, V_{n-1}, V_n$ are the present values of the 3rd, …, $(n-1)^{th}$ and n^{th} installments, then $V_3 = \frac{P}{(1+i)^2}, \mathbf{L}, V_{n-1} = \frac{P}{(1+i)^{n-2}}, V_n = \frac{P}{(1+i)^{n-1}}$.

Now, if V is the present value of the annuity due, then

$$V = V_1 + V_2 + \mathbf{L} + V_{n-1} + V_n$$

$$= P + \frac{P}{1+i} + \mathbf{L} + \frac{P}{(1+i)^{n-2}} + \frac{P}{(1+i)^{n-1}}$$

$$= \frac{P}{(1+i)^{n-1}} \times \left[1 + (1+i) + \mathbf{L} + (1+i)^{n-1}\right] = \frac{P}{(1+i)^{n-1}} \times \left[\frac{(1+i)^n - 1}{1+i-1}\right]$$

$$= \frac{P \times (1+i)}{i} \times \left[1 - (1+i)^{-n}\right] \quad \mathbf{L}\ (4.26)$$

Exercise 4.19: A person took a loan of Rs.15,000/- on condition that he would repay the loan at 10 equal installments at 4% compound interest. If each installment is paid at the end of each year, find the installment.

Solution: Here V (present value in Rupees) = 15000, n = 10, i = 0.04.

Let P (in Rupees) be the value of each installment. Then

$$V = \frac{P}{i} \times \left[1 - (1+i)^{-n}\right] \Rightarrow 15000 = \frac{P}{0.04} \times \left[1 - (1.04)^{-10}\right]$$

$$\Rightarrow \quad P \times \left[1 - (1.04)^{-10}\right] = 600 \quad \mathbf{L}\ (1)$$

Let $x = (1.04)^{10}$. Then, $\log x = 10 \times \log(1.04) = 10 \times 0.01703 = 0.1703$

This implies that $x = anti\log(0.1703) = 1.480$.

From (1), we get $P = \frac{600}{1 - 1/1.48} = \frac{600 \times 1.48}{1.48 - 1} = 1850$

Therefore value of the each installment is equal to Rs.1850/-.

Exercise 4.20: Find the present value of the annuity of Rs.2000 payable at the beginning six months for 12 years at 8% p.a. interest compounded half yearly.

Solution: This is the case of annuity due. If V is the present value of the annuity, then

$$V = \frac{P \times (1+i)}{i} \times \left[1 - (1+i)^{-n}\right] \quad \mathbf{L}\ (1)$$

Here P = 2000, i = half yearly interest on Re 1 = $\frac{8/2}{100} = 0.04$

$n = 2 \times 12 = 24$

$\therefore \quad (1) \Rightarrow V = \dfrac{2000 \times (1+0.04)}{0.04} \times \left[1-(1+0.04)^{-24}\right] = 52000 \times \left[1-(1.04)^{-24}\right]$

Let $x = (1.04)^{-24}$. Then,

$$\log x = -24 \times \log(1.04) = (-24) \times 0.017033 = -0.408792$$

$$= \bar{1}.591208 = \bar{1}.5912 \quad (approx)$$

This implies that $x = anti\log(\bar{1}.5912) = 0.3901$

Hence, $V = 52000 \times [1-0.3901] = 31714.80$

Thus the required present value = Rs.31,714.80.

3.8 LET US SUM UP

1. In this unit, we have learnt about two types of interest: simple and compound interest. We have derived formulas (3.1 to 3.7) to find simple interest, amount and principal value. In case of simple interest the interest is charged only on the borrowed money (principal) throughout the period of loan.
2. In case of compound interest, interest on the principal for first year (or first six months or first 3 months) of the period of the loan is added to the principal for the second year (or second six months or second 3 months) of the period. This process is continued for the period of the loan. We have derived formulas (3.8 to 3.22) to find compound interest, amount and the principal for yearly, half yearly and quarterly calculations.
3. We have also learnt about a very important concept which is depreciation. The value of asset like buildings, machines etc go on decreasing year after year since their life time goes on diminishing. The value of the depreciable asset at the end of its useful life time is called the scrap value. The total depreciation of the asset is equal to the difference between the original value and the scrap value.
4. Finally we have discussed another very important concept which is

Annuity. An annuity is a fixed sum of money paid periodically under some conditions. We have learnt about five types of Annuity, namely, Ordinary Annuity, Annuity Due, Deferred Annuity, Life Annuity and Perpetual Annuity. We have derived formulas (3.23 and 3.24) to find amount of Ordinary Annuity and Annuity Due respectively. Using the formulas 3.25 and 3.26, we can find the present value of Ordinary Annuity and Annuity Due respectively.

3.9 FURTHER READINGS

1. Khanna V. K., Zameeruddin Qazi & Bhambri S.K.(1995). Business Mathematics, New Delhi, Vikas Publishing House Pvt Ltd. .
2. Hazarika P.L. Business Mathematics, New Delhi. S.Chand & Co.

3.10 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (i) T, (ii) T, (iii) F, (iv) F, (v) T.

Ans to Q No 2: (i) Rs.468.75 (ii) Rs.937.50 (iii) 8% p.a. (iv) Rs.500, 8.5% p.a.

Ans to Q No 3: (i) T, (ii) T, (iii) F, (iv) T.

Ans to Q No 4: (i) Rs.2497.28 (ii) Rs.6907

Ans to Q No 5: (1) (i) T, (ii) F, (iii) T, (iv) F, (v) T.

Ans to Q No 6: (i) Rs.1593.00 (ii) Rs.18,396.20

3.11 MODEL QUESTIONS

1. A man borrowed Rs.12,800 at 8.5% per annum simple interest for 3 years 3 months. At the end of the period he repaid the money along with the interest. Find how much money he paid.

2. A man borrowed Rs.15,000 for 9 months at 8% per annum simple interest and another man borrowed some money at 12% per annum simple interest for 1 year 8 months. If the two men paid the same amount to clear their debts along with the interest, how much money did the second man borrow?
3. If the difference between the compound interest and the simple interest on a sum of money at 8% per annum interest for 2 years is Rs.200, find the sum of money.
4. In what time Rs.1250 will amount to Rs.1625 at 8% per annum simple interest? At what rate of simple interest the same sum of money will amount to Rs.1718.75 in the same period?
5. Find the compound interest on Rs.96000 for 2 years 3 months at 5% per annum interest being compounded annually.
6. What sum of money will amount to Rs.23060 in 3 years at 4.8% per annum, interest being compounded half yearly.
7. The scrap value of a machine costing Rs.120000 depreciated at 5% per annum is found to be Rs.36456. Find the usual life period of the machine.
8. Find the amount of an annuity of Rs.450 payable at the end of each year for 18 years at 4.5% compound interest per annum.
9. A man borrowed Rs.39260 and agreed to pay the amount along with interest in equal 20 installments payable at the end of each year. If rate of compound interest is 8% per annum, find the value of each installment.
10. Find the principal value to be invested at p.a. so that after 5 years the amount will be Rs.12,000 if the interest is compounded annually.
11. Find the present value of an annuity of Rs.1000 payable at the end of each year for 15 years if the rate of compounded interest is 4% p.a.

*** ***** ***

UNIT 4: BASIC ALGEBRAIC CONCEPTS

UNIT STRUCTURE

4.1 LEARNING OBJECTIVES

After going through this unit you will be able to:

- solve quadratic equations by factorizing.
- solve quadratic equations by completing the square.
- solve quadratic equations by using the quadratic formula.
- solve simultaneous linear equations in two unknowns.

4.2 INTRODUCTION

We are already familiar with linear polynomial and linear equation. The general form of linear equation of first-degree equation is ax+by+c=0 where a, b, c are real numbers. Consider the following linear model that describes the time necessary to load a van with parcels for distribution.

Time to load the van = (a x number of parcels) + b

where a represents the time per parcel and b is a constant value representing the time for ancillary parts of the job, for example, cleaning out the van each time ready for the next load.

As an example, suppose we have 12 parcels to deliver and a = 1 minute, b = 5 minute. Inserting this figure in our equation, the time required to load the van is: (1 x 12) + 5 = 17 minutes.

But there are many problems that cannot be solved by linear equations. The solutions of problems such as given below can be solved by equations which are quadratic.

1. The sum of a positive integer and its square is 20. What is the number?
2. The height of a triangle is less than the base by 2 cm and its area is 12 cm2. Find the length of the base.

Similarly, there are situations where we have to deal with two unknowns (or variables). Like linear equation in one unknown, we have linear equations in two unknowns. If we consider two linear equations in two unknowns at the same time, we have a system of simultaneous linear equations.

4.3. QUADRATIC EQUATIONS

The algebraic expressions like , x^2-4, x^2+x-6, $4x+\frac{1}{x}+4$ etc. are quadratic polynomials or second-degree polynomial in. The general form of a quadratic polynomial is ax^2+bx+c, where are real numbers and a $\neq$ 0.

An equation in which the highest power of the variable (or unknown) is two is called a quadratic equation. The equations like $x^2-4=0$, $x^2+x-6=0$, $4x+\frac{1}{x}+4=0$ etc are called quadratic equations.

The general or standard form of a quadratic equation is:

$$ax^2+bx+c=0, \text{ a, b, c are real with a} \neq 0.$$

The only requirement here is that we have a term in x^2 in the equation. We guarantee that this term will be present in the equation by requiring a $\neq$ 0. Note that it is all right if b or c or both are zero.

If *b* **is equal to zero**, the quadratic equation is called **pure quadratic equation.**

If b **is non zero,** the quadratic equation is called **mixed quadratic equation.**

4.3.1 EXAMPLES OF QUADRATIC EQUATIONS:

(a) $x^2 = 0$ is a pure quadratic equation.

(b) $x^2 -3x -28 = 0$ is a mixed quadratic equation.

(c) $(x-2)^2 = x^2 + 6$ is not a quadratic equation, since on simplification, the equation reduces to $2x+1= 0$ (which is linear).

CHECK YOUR PROGRESS

Q 1: State whether the following statements are true (T) or false (F).

(a) $\frac{x+1}{2} = \frac{2}{x}$ is not a quadratic equation. (T / F)

(b) $(x+1)(x+4) = 2x^2 + x + 7$ is a pure quadratic equation. (T / F)

(c) $(x-1)(2x+1) = x(x+5)$ is a mixed quadratic equation. (T / F)

(d) $(x+1)^3 = 2x(x^2+1)$ is not a quadratic equation. (T / F)

4.3.2 SOLUTION OR ROOT OF QUADRATIC EQUATION

The value of the variable for which both sides of the equation are equal is a root of the equation. A quadratic equation has two roots. For example $x=1, x=2$ are the roots of the quadratic equation $x^2 - 3x + 2 = 0$.

For $x=1$, $1^2 - 3\times1 + 2 = 1 - 3 + 2 = 0$.

For $x=2$, $2^2 - 3\times2 + 2 = 4 - 6 + 2 = 0$.

But $x=3$ is not a root of the equation. Because $3^2 - 3\times3 + 2 = 9 - 9 + 2 = 2 \neq 0$.

Then what are the methods of solving a quadratic equation? We now discuss it.

4.3.3 METHODS OF SOLVING QUADRATIC EQUATIONS

A. Pure quadratic equation can be expressed in the form $x^2=d$ (d is a constant). Hence the roots are $x = \sqrt{d}$ and $x = -\sqrt{d}$.

B. An mixed quadratic equation can be solved by

(i) the factorization method,

(ii) completing the square,

(iii) Quadratic formula.

4.3.3 (A) SOLUTIONS OF PURE QUADRATIC EQUATIONS

Exercise 1: Solve

(a) $x^2 = 0$

(b) $\frac{1}{2}x^2 - 3 = 0$

(c) $(x+2)(x-2) = 12.$

Solutions:

(a) $x^2 = 0 \Rightarrow x = \pm\, 0$.

Here both the solutions are equal to 0.

(b) $\frac{1}{2}x^2 - 3 = 0 \Rightarrow \frac{1}{2}x^2 = 3 \Rightarrow x^2 = 6 \Rightarrow x = \pm\sqrt{6}.$

(c) $(x+2)(x-2) = 12 \Rightarrow x^2 - 4 = 12 \Rightarrow x^2 = 16$

$\Rightarrow x = \pm\sqrt{16} \Rightarrow x = \pm\, 4.$

4.3.3 B (i) SOLUTION BY FACTORIZATION

In this section we will discuss the factorization method to solve quadratic equations. To do this we will need the following fact.

> If *p* and *q* are two real numbers such that *pq* = 0 then either *p* = 0 or *q* = 0 or both *p* and *q* are equal to zero.

This fact is called the **zero factor property** or **zero factor principle.** All the fact says is that if a product of two terms is zero then at least one of the terms is equal to zero.

Remark: Notice that this fact will ONLY work if the product is equal to zero. Consider the following product:

pq = 6

In this case there is no reason to believe that either *p* or *q* will be 6. We could have *p* = 2 and *q* = 3 for instance. So, do not misuse this fact!

To solve a quadratic equation by factorizing we first must move all the terms over to left hand side of the equation. Doing this serves two purposes. First, it puts the quadratics into a form such that the left-hand side can be expressed as a product of two linear factors. Secondly, and probably more importantly, in order to use the zero factor property we MUST have a zero on one side of the equation. If we don't have a zero on one side of the equation we won't be able to use the zero factor property.

Let us take a look at a couple of examples.

Exercise 2: Solve by factorization method:

(i) $x^2 - x = 12$

Solution: First get everything on the left hand side of the equation. Then,

$$x^2 - x - 12 = 0$$
$$\Rightarrow x^2 - 4x + 3x - 12 = 0$$
$$\Rightarrow x(x-4) + 3(x-4) = 0$$
$$\Rightarrow (x-4)(x+3) = 0.$$

Now at this point we have got a product of two terms that is equal to zero. By zero factor property, we have

Either $x - 4 = 0$ or $x + 3 = 0.$

Hence the roots of the given quadratic equation are 4 and-3.

(ii) $x^2 + 40 = -14x$

Solution: As with the first one we first get everything on the left-hand side of the equal sign and then express it as a product of two factors.

$$x^2 + 14x + 40 = 0$$
$$\Rightarrow x^2 + 4x + 10x + 40 = 0$$
$$\Rightarrow x(x+4) + 10(x+4) = 0$$
$$\Rightarrow (x+4)(x+10) = 0.$$

Hence applying the zero factor property, the roots of the given quadratic equation are –4 and –10.

(iii) $y^2 + 12y + 36 = 0$

Solution: In this case we already have zero on one side and so we don't need to do any manipulation to the equation. We just express it as a product of two factors. Also, do not get excited about the fact that we now have y's in the equation. We won't always be dealing with x's so do not expect to see them always. Let us now factorize the left hand-side of the equation:

$$y^2 + 12y + 36 = 0$$
$$\Rightarrow (y+6)^2 = 0$$
$$\Rightarrow (y+6)(y+6) = 0$$

In this case we've got a perfect square on the left-hand side. We broke up the square to denote that we really do have an application of the zero factor property. However, we usually don't do that. We usually will go straight to the answer from the squared part.

The solution to the equation in this case is,

$$y = -6$$

and it has come from the product of two same linear factors on the left hand side. This is the case of having two equal roots.

(iv) $\dfrac{1}{x+1} = 1 - \dfrac{5}{2x-4}$.

Solution: After simplification, the given equation reduces to $2x^2 - 9x - 5 = 0$. We can factorize the left-hand side of this equation as $(2x+1)(x-5) = 0$. So, the two solutions to this equation are 5 and $-\dfrac{1}{2}$.

(v) $2x^4 - 11x^2 + 12 = 0$

Solution: The given equation is not a quadratic equation. It is a fourth degree equation or bi-quadratic equation. We can reduce it to a quadratic equation by substituting y for x^2. Thus the given equation reduces to $2y^2 - 11y + 12 = 0$ and factorizing, we get $(y-4)(2y-3) = 0$. This implies either y - 4 = 0 or 2y - 3 = 0. That is $x^2 = 4$ or 2 $x^2 = 3$ and hence the solutions are:

$$x = 2, -2, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}$$

4.3.3. B(ii) SOLUTION BY COMPLETING THE SQUARE

1. First method: Let the quadratic equation be:

$$ax^2+bx+c=0,\ a\neq 0.$$

$\Rightarrow ax^2+bx=-c$, [Transposing to the right hand side]

$\Rightarrow x^2+\frac{b}{a}x=-\frac{c}{a}$, [Dividing both sides by a, note that this is possible since $a\neq 0.$]

$$\Rightarrow x^2+2\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}\text{, [Adding }\left(\frac{b}{2a}\right)^2\text{ to both sides]}$$

$$\Rightarrow \left(x+\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$$

$$\Rightarrow \left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$\Rightarrow \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}\text{, [Taking square root on both sides]}$$

$$\Rightarrow x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Hence the roots are: $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$.

2. Second method: Let the quadratic equation be:

$$ax^2+bx+c=0,\ a\neq 0.$$

$$\Rightarrow ax^2+bx=-c$$

$\Rightarrow 4a(ax^2+bx)=-4ac$, [Multiplying both sides by 4a]

$$\Rightarrow 4a^2x^2+4abx=-4ac$$

$\Rightarrow (2ax)^2+2.2ax.b+b^2=b^2-4ac$, [Adding to both sides]

$$\Rightarrow (2ax+b)^2=b^2-4ac$$

$$\Rightarrow 2ax + b = \pm\sqrt{b^2 - 4ac}$$

$$\Rightarrow 2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence the roots are: $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Remark: The famous Hindu Mathematician Shridhar Acharjya gave this method of completing square multiplying by 4a .

Exercise 3: Solve $7x^2 - 15x + 2 = 0$ by the method of completing square.

Solution: First method:

$$7x^2 - 15x + 2 = 0 \Rightarrow 7x^2 - 15x = -2$$

$$\Rightarrow x^2 - \frac{15}{7}x = -\frac{2}{7}$$

$$\Rightarrow x^2 - 2.\frac{15}{14}x + \left(\frac{15}{14}\right)^2 = -\frac{2}{7} + \left(\frac{15}{14}\right)^2$$

$$\Rightarrow (x - \frac{15}{14})^2 = -\frac{2}{7} + \left(\frac{15}{14}\right)^2$$

$$\Rightarrow (x - \frac{15}{14})^2 = \frac{225}{196} - \frac{2}{7}$$

$$\Rightarrow (x - \frac{15}{14})^2 = \frac{225 - 56}{196} = \frac{169}{196}$$

$$\Rightarrow x - \frac{15}{14} = \pm\frac{13}{14}$$, [Taking square roots on both sides]

$$\Rightarrow x = \frac{15}{14} \pm \frac{13}{14}.$$

$$\Rightarrow x = \frac{15}{14} + \frac{13}{14} \text{ and } x = \frac{15}{14} - \frac{13}{14}$$

This implies that $x = \frac{28}{14} = 2$ and $x = \frac{2}{14} = \frac{1}{7}$.

Hence the required roots of the given equation are: 2 and $\frac{1}{7}$.

Second method:

$$7x^2 - 15x + 2 = 0$$

$$\Rightarrow 7x^2 - 15x = -2$$

$$\Rightarrow 28(7x^2 - 15x) = 28(-2) \text{ [Multiplying both sides by 4a = 28]}$$

$$\Rightarrow (14x)^2 - 2\times 14x \times 15 + (15)^2 = (15)^2 - 56$$

$$\Rightarrow (14x - 15)^2 = 225 - 56 = 169 = 13^2$$

$$\Rightarrow 14x - 15 = \pm 13$$

$$\Rightarrow x = \frac{15+13}{14}, \quad x = \frac{15-13}{14}$$

$$\Rightarrow x = 2, \quad x = \frac{1}{7}$$

Hence the required roots of the given equation are: 2 and $\frac{1}{7}$.

4.3.3. B(iii) SOLUTION BY USING QUADRATIC FORMULA

We have seen in B(ii) that the solution of the quadratic equation

$$ax^2 + bx + c = 0, a \neq 0$$

are given by the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This formula is known as **quadratic formula**. We now solve two problems to know how to solve quadratic equations by this method.

Exercise 4: Solve by using quadratic formula:

(i) $2x^2 + 3 = 7x$

(ii) $\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3} = 0$

Solution: (i) The given equation reduces to the standard form:

$2x^2 - 7x + 3 = 0$ and comparing this with $ax^2 + bx + c = 0$ we get a = 2, b = - 7, and c = 3. Using quadratic formula, we find

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4\times 2\times 3}}{2\times 2} = \frac{7 \pm 5}{4}$$

Hence the required roots are: 3 and $\frac{1}{2}$.

(iii) $\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}=0$

$\Rightarrow (x-2)(x-3)+(x-1)(x-3)+(x-1)(x-2)=0$

$\Rightarrow 3x^2-12x+11=0.$

This equation is in standard form with a = 3, b = - 12, c = 11.

Using quadratic formula, we get

$$x=\frac{-(-12)\pm\sqrt{(-12)^2-4\times3\times11}}{2\times3}=\frac{12\pm\sqrt{12}}{6}=\frac{6\pm\sqrt{3}}{3}$$

Hence the roots are: $\frac{6+\sqrt{3}}{3}$, $\frac{6-\sqrt{3}}{3}$

4.3.4. NATURE OF ROOTS OF QUADRATIC EQUATIONS

We have seen that the roots of the quadratic equation $ax^2+bx+c=0, a\neq 0$

are given by the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Now, the following cases may arise:

Case 1: If $b^2-4ac=0$, then $x=\frac{-b\pm 0}{2a}=-\frac{b}{2a}$.

Therefore, the roots are real and equal.

Case 2: If $b^2-4ac>0$, then the roots are real and distinct.

The roots are: $\frac{-b+\sqrt{m}}{2a}$, $\frac{-b-\sqrt{m}}{2a}$, where

$m=b^2-4ac>0\cdot$

Case 3: If $b^2-4ac<0$, then we can not find any real number whose square is b^2-4ac. Hence, the roots are not real.

Thus we have seen that b^2-4ac determines whether the roots of the quadratic equation $ax^2+bx+c=0$ real or not, whether roots are distinct or not. For this reason, b^2-4ac is called the discriminant of the quadratic equation. You should notice that we can find the nature of the roots without solving the quadratic equation.

CHECK YOUR PROGRESS

Q 2: State whether the following statements are true (T) or false (F).

(i) The roots of $x^2 + 2x + 1 = 0$ are equal and real. (T / F)

(ii) If the roots of a quadratic equation are equal, then they are always real. (T / F)

(iii) The roots of $x^2 + x + 1 = 0$ are real. (T / F)

(iv) The roots of $2x^2 + 4x + 1 = 0$ are distinct and real. (T / F)

Q 3: Solve the following quadratic equations by factorization method, completing the squares, and using quadratic formula, respectively.

(i) $x^2 - 7x + 10 = 0$ (ii) $x^2 + 9x + 14 = 0$

(iii) $x^2 - 3x - 10 = 0$ (iv) $x - 6 + \frac{9}{x} = 0$

4.4 SIMULTANEOUS EQUATION WITH TWO UNKNOWNS (OR VARIABLES)

A linear equation in two variables differs from a linear equation in one variable in having two unknowns (or variables) instead of one unknown. The **general or standard form** of a linear equation in two unknowns is:

$$\boxed{ax + by + c = 0,\ a,\ b,\ c\ \textbf{are real with}\ a \neq 0, b \neq 0.}$$

Examples:

(1) $x + y = 0$ (2) $x + y + 1 = 0$

The pair of values, one for and the other for which when substituted in the equation, make both sides of the equation equal is called a solution of the equation.

For example: $x = 1,\ y = 3$ is a solution of the equation $2x + y = 5$.

The graph of a linear equation in two unknowns is a straight line. Every solution of the equation is a point on the straight line and conversely the co-ordinates of every point on the straight line represented by the equation is a solution of the equation. Here we shall discuss the solution of a linear equation of a pair of linear equations in two unknowns or simply simultaneous equations in two unknowns.

A pair of linear equations of the form

$$\boxed{\begin{array}{c} a_1x+b_1y+c_1=0 \text{ and } a_2x+b_2y+c_2=0, \text{ where} \\ a_1, b_1, c_1, a_2, b_2, c_2 \text{ are real numbers and} \\ a_1 \neq 0, b_1 \neq 0, a_2 \neq 0, b_2 \neq 0 \end{array}}$$

is known as a system of simultaneous equations in two unknowns *x* and *y*.

There are two categories of system of simultaneous equations. These are:

(i) **System of simultaneous homogeneous equations:** If both c_1 and c_2 are zero, then such a system of simultaneous equations is called a system of simultaneous homogeneous equations.

Examples:

(a) $\begin{array}{l} x+y=0 \\ 2x+y=0 \end{array}$ (b) $\begin{array}{l} x-y=0 \\ x+2y=0 \end{array}$

(ii) **System of simultaneous non-homogeneous equations:** If $a_1, b_1, c_1, a_2, b_2, c_2$ are all non-zero real numbers, then such a system of simultaneous equations is called a system of simultaneous non-homogeneous equations.

Examples:

(a) $\begin{array}{l} x+y+1=0 \\ 2x+y-1=0 \end{array}$ (b) $\begin{array}{l} x-y+5=0 \\ x+2y+1=0 \end{array}$

Now, we shall learn about the solution of a system of simultaneous equations. The pair of values, one for x and the other for y which satisfy both the equations is called the solution of the system. For example, $x=2,\ y=1$ is the solution of the simultaneous equation: $2x+3y=7$ and $5x-2y=8$.

4.4.1. CONDITION FOR CONSISTENCY

Let us consider the system of simultaneous equations:

$$a_1x+b_1y+c_1=0 \quad \ldots (1)$$

$$a_2x+b_2y+c_2=0 \quad \ldots (2)$$

Multiplying equation (1) by b_2 and equation (2) by b_1, we get

$$a_1b_2x+b_1b_2y+c_1b_2=0 \quad \ldots (3)$$

$$a_2b_1x + b_1b_2y + b_1c_2 = 0 \ \ldots(4)$$

(3) - (4) gives,

$$(a_1b_2 - a_2b_1)x = b_1c_2 - b_2c_1 \ldots(5)$$

Similarly, multiplying equation (1) by a_2 and equation (2) by a_1 and subtracting we get,

$$(a_1b_2 - a_2b_1)y = a_2c_1 - a_1c_2 \ldots(6)$$

CASE 1: If $a_1b_2 - a_2b_1 \neq 0$, then from (5) and (6), we get

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} \ldots(7)$$

Thus the system is consistent and has the unique solution given by (7). Thus if $a_1b_2 - a_2b_1 \neq 0$, that is, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then the straight lines represented by equation (1) and (2) intersect at a point having co-ordinates

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right)$$

For example, consider the system of simultaneous equations:

$$x + 2y = 4 \ \ldots(\text{i})$$

$$3x - y = 5 \ldots(\text{ii})$$

Then $a_1b_2 - a_2b_1 = 1 \times (-1) - 3 \times 2 = -7 \neq 0$ and the lines represented by (i) and (ii) intersect at the point (2, 1) as shown in the figure 1.

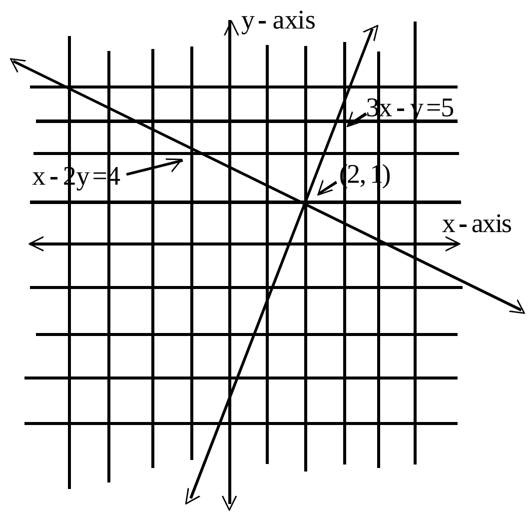


Figure 1

CASE 2: If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then let $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k$. Clearly k is non zero. We get $a_1 = ka_2, b_1 = kb_2, c_1 = kc_2$. Putting the values of a_1, b_1, c_1 in equation (1), we get $k(a_2x + b_2y + c_2) = 0$.

Thus the system

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

is equivalent to the system

$$k(a_2x + b_2y + c_2) = 0$$
$$a_2x + b_2y + c_2 = 0$$

Clearly every solution of the first system is a solution of the other and vice-versa. Since k is non-zero, the two equations (1) and (2) represent the same equation. Therefore the system has infinitely many solutions. Geometrically it means that the two straight lines represented by the two equations are coincident.

For example, consider the system of simultaneous equations:

$$6x - 3y = 12$$
$$2x - y = 4$$

The two lines represented by the above two equations which are coincident are shown in the figure 2.

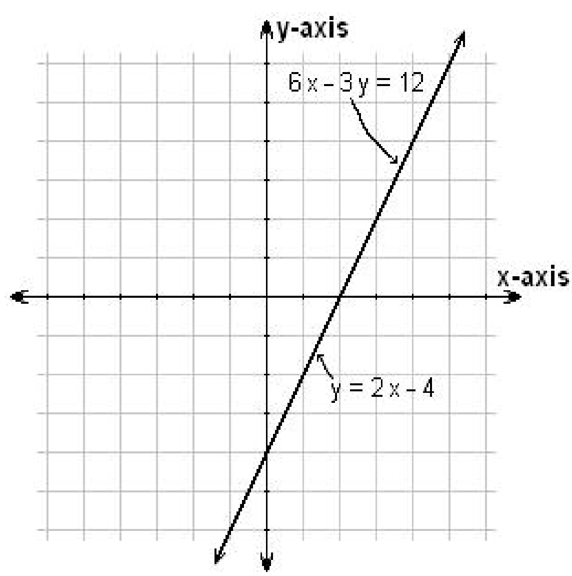


Figure 2.

CASE 3: If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the system has no solution. In this case the straight lines represented by the two equations become parallel and hence they do not intersect at any point. For example, consider the system of simultaneous equations:

$$3x - 2y = 2$$
$$3x - 2y = -2.$$

The two lines represented by the above equations are parallel to each other as shown in the figure 3.

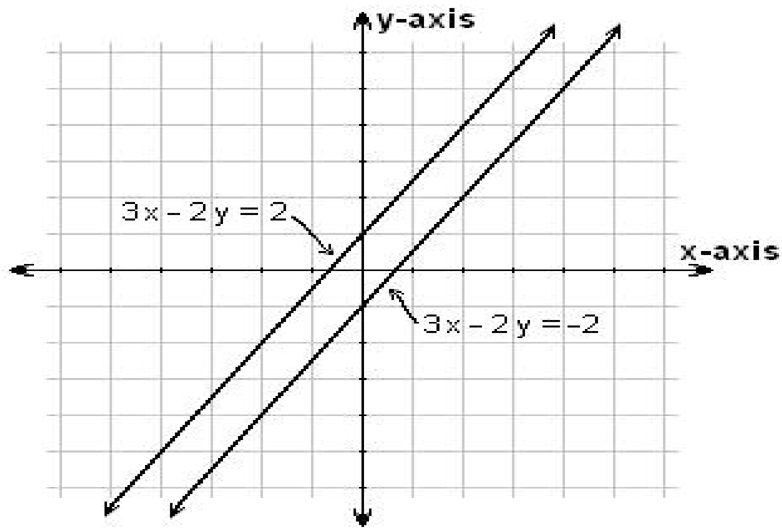


Figure 3.

Now, let us solve some problems.

Exercise 5: In each of the following systems of equations determine whether the system has unique solution, infinitely many solutions or no solution.

(i) $2x - y = 7$ (ii) $3x - 2y = 5$

$x + 5y = -2$ $\frac{9}{2}x - 3y - \frac{15}{2} = 0$

(iii) $5x - 2y = 12$

$\frac{15}{2}x - 3y = 24$

Solutions:

(i) $2x - y = 7,\ x + 5y = -2 \Rightarrow 2x - y - 7 = 0,\ x + 5y + 2 = 0$

Here, $\frac{a_1}{a_2} = \frac{2}{1} = 2, \quad \frac{b_1}{b_2} = \frac{-1}{5}, \quad \frac{c_1}{c_2} = \frac{-7}{2}$

Since $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, the system has unique solution.

(ii) $3x - 2y = 5, \quad \frac{9}{2}x - 3y - \frac{15}{2} = 0$

$\Rightarrow 3x - 2y - 5 = 0, \quad 9x - 6y - 15 = 0$

Here, $\frac{a_1}{a_2} = \frac{1}{3}, \quad \frac{b_1}{b_2} = \frac{1}{3}, \quad \frac{c_1}{c_2} = \frac{1}{3}$.

Since $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, the system has infinitely many solutions.

(iii) Here, $\frac{a_1}{a_2} = \frac{5}{15} = \frac{1}{3}, \quad \frac{b_1}{b_2} = \frac{-2}{-6} = \frac{1}{3}, \quad \frac{c_1}{c_2} = \frac{-12}{-48} = \frac{1}{4}$.

Since $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, the system has no solution.

Exercise 6: Find the value of k for which the system of equations

$$(k-3)x + 3y = k$$
$$kx + ky = 12$$

has infinitely many solutions. Also find some values of k for which the system has a unique solution.

Solution: For infinitely many solutions to exist, the system must satisfy $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$. That is, the given system must satisfy

$$\frac{k-3}{k} = \frac{3}{k} = \frac{-k}{-12} \quad \ldots(1)$$

Now, $\frac{3}{k} = \frac{-k}{-12} \Rightarrow k^2 = 36 \Rightarrow k = \pm 6$.

Verify that, k = 6 satisfy (1) and k = - 6 does not satisfy (1). Hence the system has infinitely many solutions if k = 6.

Second part: The given system has a unique solution if $\frac{k-3}{k} \neq \frac{3}{k}$.

That is, if $k^2 \neq 6k$. Some possible values of k are 1, 2, 3 etc.

4.4.2. METHOD OF SOLUTIONS OF SIMULTANEOUS EQUATIONS:

Using algebraic method and graphical method, one can solve a system of simultaneous equation in two variables.

(A) Algebraic methods are:

(i) Elimination by equating the coefficients.

(ii) Elimination by substitution.

(iii) Cross multiplication.

(B) Graphical method: In this method graphs of the two equations are drawn using the same axes of co-ordinates and the same unit.

A (i). **Method of elimination by equating the co-efficients:**

Consider the equations:

$$a_1x + b_1y + c_1 = 0 \quad \ldots (1)$$

$$a_2x + b_2y + c_2 = 0 \quad \ldots (2)$$

As shown in section 1.4.1, multiplying (1) and (2) by a2 and a1 respectively, the co-efficients of x in both the equations can be made equal. After subtracting we get an equation in y only from which we can get the value of y. Putting this value in equation (1) or (2), the value of x can be obtained. Instead of equating the co-efficients of x, we can make the co-efficients of y in both the equations equal and proceeding as above, the solution can be obtained.

Exercise 7: Using method of elimination by equating the coefficients, solve the following system.

$$7x + 5y = 8 \quad \ldots (1)$$

$$2x - 3y = -11 \quad \ldots (2)$$

Solution: Multiplying (1) by 3 and (2) by 5, we get

$$21x + 15y = 24 \quad \ldots (3)$$

$$10x - 15y = -55 \quad \ldots (4)$$

Adding (3) and (4), we get $21x + 10x = -31 \Rightarrow x = -1$.

Putting $x = -1$ in (1), we obtain $y = 3$.

Hence the solution is $x = -1,\ y = 3$.

A(ii). **Method of elimination by substitution:**

Consider the equations:

$$a_1 x + b_1 y + c_1 = 0 \quad \ldots (1)$$

$$a_2 x + b_2 y + c_2 = 0 \quad \ldots (2)$$

From (1), we can express y in terms of x and in doing so we get

$$y = \frac{-a_1 x - c_1}{b_1} \quad \ldots (3)$$

Substituting this value of y in (2), we get

$a_2 x + b_2\left(\dfrac{-a_1 x - c_1}{b_1}\right) + c_2 = 0$ and solving this we get the value of x and then putting this value of x in (3), we can find the value of y.

Instead of expressing y in terms of x from (1), we can express x in terms of y and proceed as above to get the solution.

Exercise 8: Using the method of elimination by substitution, solve the following system.

$$\frac{3}{2}x - \frac{5}{3}y = -2 \quad \ldots (1)$$

$$\frac{1}{3}x + \frac{1}{2}y = \frac{13}{6} \quad \ldots (2)$$

Solution: Multiplying (1) and (2) by 6, we simplify the equations and we get

$$9x - 10y = -12 \quad \ldots (3)$$

$$2x + 3y = 13 \quad \ldots (4)$$

From (4), $y = \dfrac{13 - 2x}{3} \quad \ldots (5)$ and putting this value of y in (3),

we get,

$$9x - 10\left(\frac{13 - 2x}{3}\right) = -12$$

$$\Rightarrow 27x - 130 + 20x = -3$$

$$\Rightarrow 47x = 130 - 36 = 94$$

$$\Rightarrow x = 2$$

Putting this value of x in (5), we get y = (13 - 4) / 3 = 3.

Hence x = 2, y = 3 is the solution.

A(ii). **Method of cross multiplication:** Let us consider the following system of equations:

$$a_1x + b_1y + c_1 = 0 \;\dots\text{(1)}$$

$$a_2x + b_2y + c_2 = 0 \;\dots\text{(2)}$$

Multiplying (1) by b_2 and (2) by b_1 and subtracting we get

$$(a_1b_2 - a_2b_1)x + (b_2c_1 - b_1c_2) = 0 \quad \Rightarrow x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \dots \text{(3)}$$

Similarly multiplying (1) by a_2 and (2) by a_1 and subtracting we get

$$(a_1b_2 - a_2b_1)y + (a_1c_2 - a_2c_1) = 0 \quad \Rightarrow y = \frac{c_1a_2 - a_2c_1}{a_1b_2 - a_2b_1} \dots \text{(4)}$$

From (3) and (4) we get

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1} \;\dots\text{(5)}$$

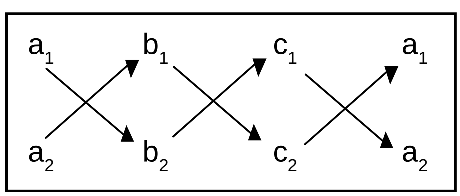
Important Steps:

(i) Write the two equations in the standard form as:

$$a_1x + b_1y + c_1 = 0 \;\dots\text{(1)}$$

$$a_2x + b_2y + c_2 = 0 \;\dots\text{(2)}$$

(ii) Write the co-efficients as:



(iii) $$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

(iv) $$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Example 9: Solve the following system of equations by cross multiplication:

$$2x + 3y = 11 \quad \dots(1)$$

$3x + 2y = 10 \quad ...(2)$

Solution: (1) and (2) can be written as:

$2x + 3y - 11 = 0$

$3x + 2y - 10 = 0$

Arranging the co-efficients as:

$$\begin{array}{cccc} 2 & 3 & -11 & 2 \\ 3 & 2 & -10 & 3 \end{array}$$

Now, applying cross multiplication method, we get

$$\frac{x}{3\times(-10) - 2\times(-11)} = \frac{y}{(-11)\times 3 - (-10)\times 2} = \frac{1}{2\times 2 - 3\times 3}$$

This implies, $\dfrac{x}{-8} = \dfrac{y}{-13} = \dfrac{1}{-5} \Rightarrow x = \dfrac{8}{5}, \ y = \dfrac{13}{5}$

Hence $x = \dfrac{8}{5}, \ y = \dfrac{13}{5}$ is the solution.

(C) Graphical Method: In this method, we draw the graphs of two equations. The co-ordinates of the point of their intersection give the solution of the system of equations. The method will be clear from the following solved problem.

Example 10: Solve graphically:

$x - y + 1 \quad = 0 \ ...(1)$

$x + 2y + 4 = 0 \ ...(2)$

Solution: To draw the graph of (1) and (2), we find points on the straight lines represented by (1) and (2).

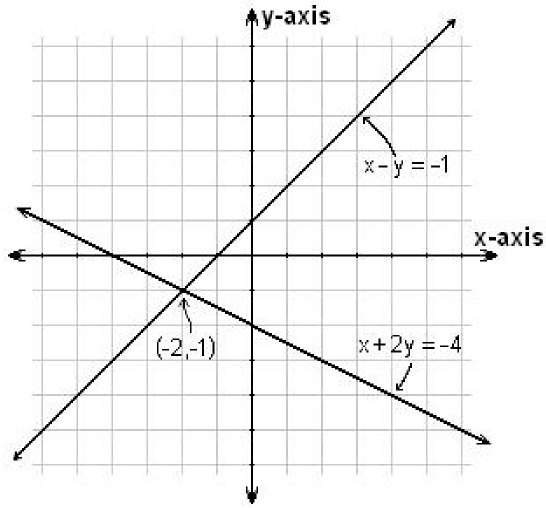
Three points on the straight line (1) :

x	0	1	-2
y	1	2	-1

Three points on the straight line (2) :

x	0	-2	-4
y	-2	-1	0

The graphs of the two straight lines are shown below:



Remember that while drawing the graphs of the two straight lines, we take the same x-axis and y-axis. Taking the point of intersection of the two axes as origin and using the same scale, we draw the lines. The point of intersection of these two lines is the solution of the given system of equations (1) and (2). In the graph, we see that the point of intersection of the two lines is
(-2, -1) and hence the solution is x = -2, y = -1.

CHECK YOUR PROGRESS

Q 4: What are the different categories of system of simultaneous equations in two unknowns?

Q 5: What is the condition for the existence of unique solution of a system of simultaneous equations?

Q 6: What is the condition for the existence of infinitely many solutions of a system of simultaneous equations?

Q 7: When does a system of simultaneous equations have no solutions?

Q 8: State whether the following statements are true (T) or false (F)

(i) Two non-parallel straight lines intersect at a single point. (T / F)

(ii) If the straight lines represented by the equations of a system of simultaneous linear equations are different parallel lines, there are infinitely many solutions of the system. (T / F)

(iii) The system $x + y = 0,\ x + y = 1$ has no solution. (T / F)

Q 9: Solve the following system of linear equations by

(i) cross multiplication:

(ii) the method of elimination by substitution

(iii) the method of elimination by equating the coefficients

(iv) graphical method.

$$2x - 3y - 1 = 0$$
$$3x + 2y - 3 = 0$$

4.5 LET US SUM UP

1. In this unit, we have learnt about two different types of quadratic equations, namely, pure and mixed quadratic equations. We have learnt how to solve pure quadratic equations. Also, we have seen three different methods to solve mixed quadratic equations and these are the factorization method, method of completing the square, and using the quadratic formula. We have solved a good number of problems using these methods.
2. We have also learnt about the discriminant of a quadratic equation. From the discriminant we get information about the nature of the roots. If the discriminant is zero, then the roots are real and equal, if the discriminant is positive, then the roots are real and distinct, otherwise the roots are not real.
3. We have discussed two categories of simultaneous linear equations in two variables, namely homogeneous and non-homogeneous. We have also discussed about the consistency of simultaneous linear equations. There are three possibilities, namely, the system of equations has a unique solution, the system has infinitely many solutions, and the system does not have any solutions. We derived three conditions to verify the above possibilities. These three possibilities are also shown in graphs.

4. Finally, we have learnt about different methods to solve system of simultaneous linear equations. There are three algebraic methods, namely, method of elimination by equating the coefficients, method of elimination by substitution, and method of cross multiplication. The graphical method is also discussed by drawing graphs of a system of simultaneous equations.

4.6 FURTHER READING

1. Khanna V. K., Zameeruddin Qazi & Bhambri S.K.(1995). Business Mathematics, New Delhi, Vikas Publishing House Pvt Ltd. .
2. Hazarika P.L. Business Mathematics, New Delhi. S.Chand & Co.

4.7 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1: (a) F, (b) F, (c) T, (d) T

Ans to Q No 2: (i) T, (ii) T, (iii) F, (iv) T,

Ans to Q No 3: (i) 5, 2, (ii) -2 , - 7, (iii) -2, 5, (iv) 3, 3.

Ans to Q No 4: Homogeneous and non-homogeneous.

Ans to Q No 5: $a_1b_2 - a_2b_1 \neq 0$

Ans to Q No 6: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$,

Ans to Q No 7: $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Ans to Q No 8: T, F, T

Ans to Q No 9: (i), (ii), (iii), (iv) $\frac{11}{13}, \quad \frac{3}{13}$

4.8 MODEL QUESTIONS

1. What is a quadratic equation? Write down the standard form of a quadratic equation.
2. What is a pure quadratic equation? Under what condition the equation $px^2 + qx + r = 0,\ p \neq 0$ becomes a pure quadratic equation?
3. What is the highest integral value of 'k' for which the quadratic equation $x^2 - 6x + k = 0$ have two real and distinct roots? For which value of k the given equation has equal roots.
4. If one of the roots of the quadratic equation $x^2 + mx + 24 = 0$ is $\frac{3}{2}$, then what is the value of m?
5. For what value of 'm' will the quadratic equation $x^2 - mx + 4 = 0$ have real and equal roots?
6. Solve the following quadratic equations by method of factorization.

   (i) $x^2 - 5x + 6 = 0$ (ii) $3x - 4 = \sqrt{2x^2 - 3x + 2}$

   (iii) $3x^2 + 10x + 3 = 0$ (iv) $\frac{x+3}{x+2} + \frac{x-3}{x-2} = \frac{2x-3}{x-1}$
7. Solve the following quadratic equation by completing square.

   (i) $2x^2 + 3x - 1 = 0$ (ii) $x^2 + 10x + 24 = 0$
8. Solve the quadratic equations 6(i) - 6(iv) and 7(i) - 7(ii) using quadratic formula.
9. What do you mean by the consistency of a system of simultaneous equations?
10. Using method of elimination by equating the coefficients, solve the following system.

$$\frac{1}{3}x - \frac{1}{2}y = 5$$

$$\frac{1}{2}x + \frac{1}{3}y = -\frac{7}{6}$$

11. Using the method of elimination by substitution, solve the following system.

$5x - 6y + 18 = 0$

$3x + 8y + 5 \quad = 0$

12. Solve the following system of equations by cross multiplication:

$2x - 3y - 19 = 0$

$3x + 2y - 9 \quad = 0$

13. Solve the system of equations given in Q10 to Q12 by graphical method.

*** ***** ***

UNIT 5: SEQUENCE AND SERIES

UNIT STRUCTURE

5.1 LEARNING OBJECTIVES

After going through this unit you will be able to:

- define Arithmetic and Geometric progression.
- explain Arithmetic and Geometric series.
- know about Arithmetic and Geometric means.

5.2 INTRODUCTION

We can write numbers following certain patterns. In this unit, we shall study two very important patterns of numbers, namely, arithmetic progression and geometric progression. We shall also study arithmetic and geometric series, arithmetic and geometric means. Thus, you will able to understand through this unit, the relation between arithmetic and geometric means.

5.3 ARITHMETIC PROGRESSION AND SERIES

Let us consider the following succession of numbers:

1. 1, 2, 3, 4, …
2. 5, 8, 11, 14, …

We observe that the succeeding numbers can be obtained by adding 1 in (1) and by adding 3 in (2). Such type of succession of numbers is called a sequence and each number is called a term of the sequence.

5.3.1 Arithmetic Progression

If in a succession of numbers each term is obtained by adding a fixed number to the preceding term except the first, then it is called an **Arithmetic Progression**. The fixed number is called the Common Difference (C.D.) of the A.P.

If an A.P. contains finite number of terms, then it is called finite A.P. and if the A.P. contains infinite number of terms, then it is called an infinite A.P.

Remark: The C.D. of an A.P. may be positive, negative or zero.

The first, second, third terms; etc of an A.P. are denoted by t_1, t_2, t_3,… etc, respectively.

Examples:

1. 1, 2, 3, … , 20 is a finite A.P. with first term = 1 and C.D. = 1.
2. 5, 8, 11, … is an infinite A.P. with first term = 5 and C.D. = 3.
3. 12, 8, 4, ... , -8 is a finite A.P. with first term 12 and C.D. = - 4.

Remark: C.D. of an A.P. = second term - first term

$$= t_2 - t_1$$

We shall usually denote the first term t_1 of an A.P. by a and C.D. by

d. Then the A.P. may be written as

$$\boxed{a, a+d, a+2d, a+3d, \ldots}$$

Hence the *n*th term of the A.P. is a+(n-1)d, and it is dented by tn.

Examples: (i) The first term and C.D. of the A.P. 4, 10, 16, 22, …. are 4 and 6 respectively. Thus the *n*th term of the A.P. is:

$t_n = 4+(n-1)6 = -2+6n = 2(3n-1)$

(ii) The first term and C.D. of the A.P. 1, $\frac{5}{3}$, $\frac{7}{3}$, 3, … are 1 and

$\frac{5}{3}-1=\frac{2}{3}$ respectively. Thus the *n*th term of the A.P. is:

$. t_n = 1+(n-1)\frac{2}{3} = \frac{1}{3}+\frac{2n}{3} = \frac{1}{3}(1+2n)$

Exercise 1: Which term of the A.P. 16, 11, 6, …. is -79?

Solution: Here the first term and the C.D. = -5. Suppose that -79 is the *n*th term of the A.P. Then,

$16+(n-1)\times(-5) = -79 \Rightarrow 5n = 21+79 = 100 \Rightarrow n = 20$

Thus, -79 is the 20^{th} term of the A.P.

Exercise 2: The 9th and 15th terms of an A.P. are 39 and 63 respectively. Find the 21st term?

Solution: In this problem, we see that the 9th and the 15th terms can be written as $a+8d$ and $a+14d$, where '*a*' is the first term and '*d*' is the C.D. Equating $a+8d$ and $a+14d$ to 39 and 63 respectively, we get $a+8d = 39, \quad a+14d = 63$

Solving these two equations, we find that $a = 7, \quad d = 4.$

Hence the 21st term $a+20d = 7+80 = 87.$

5.3.2 Properties of an A.P.

The following properties of an A.P. can be easily established.

(1) If each term of an A.P. is increased (or decreased) by a number, the resulting sequence becomes an A.P. with the same common difference. Let each term of the A.P. a, a + d, a + 2d, a + 3d… be increased by a number k. Therefore the resulting sequence is:

a + k, a + d + k, a + 2d + k, a + 3d + k,…

The above new sequence of numbers is an A.P. with first term a + k and C.D. = (a + d + k) - (a + k) = d.

(2) If each term of an A.P. is multiplied (or divided) by a nonzero number k, the resulting sequence is again an A.P. with c.d. equal to k times (or $\frac{1}{k}$ times) of the original A.P. Let us multiply each term of the A.P. a, a + d, a + 2d, a + 3d… by the non-zero number k. Therefore the resulting sequence is:

ka, k(a + d), k(a + 2d), k(a + 3d),…

The above new sequence of numbers is an A.P. with first term ka and C.D. = k(a + d) - ka = kd.

5.3.3 Arithmetic Series

If the terms $t_1, t_2, t_3, \ldots, t_n, \ldots$ of an A.P. are added in the given order then $t_1 + t_2 + t_3 + \ldots + t_n + \ldots$ is called an **Arithmetic Series or A.P. series.**

Examples:

(i) 1+ 2 + 3 + … + 20 is an A.P. series.

(ii) 5 + 8 + 11 + … + 35 is an A.P. series.

5.3.4 Sum of first n terms of an A.P. Series

Let us consider the A.P. series with terms:

$$a + (a + d) + (a + 2d) + \ldots + \{a + (n-1)d\}.$$

and let its sum be S_n. Rewriting the terms in reverse order, we get

$$S_n = \{a + (n-1)d\} + \{a + (n-2)d\} + \ldots + (a + d) + a.$$

Adding these two series term by term, we get

$$2S_n = \{2a + (n-1)d\} + \{2a + (n-1)d\} + \ldots + \{2a + (n-1)d\}$$

$$= n\{2a + (n-1)d\} = n\{a + [a + (n-1)d]\} = n(a + l)$$

$$\Rightarrow S_n = \frac{n}{2}\{2a + (n-1)d\} = \frac{n}{2}[a + l]$$

where *l* is the *n* th term of the series.

Exercise 3: Find the following sums:

(1) $(-9)+(-3)+3+\ldots$ up to 25th term.

(2) $4+1+(-2)+\ldots+(-53)$.

Solution: (1) Here the first term $a=-9$, C.D. $d=6,\ n=25$.

Hence the required sum is: $S_{25}=\dfrac{25}{2}\{2\times(-9)+(25-1)\times 6\}=1575$.

(2) Here we know the first term $a=4$, and C.D. $d=-3$ and the n th term $l=-53$, but we do not know the value of n. To find the sum, we must know this value.

Now, the n th term $l=a+(n-1)d=4+(n-1)\times(-3)$ which is equal to –53. That is $4+(n-1)\times(-3)=-53$. Solving for n, we find that n = 20. Hence the required sum is: $S_{20}=\dfrac{20}{2}\{4+(-53)\}=-490$.

5.3.5 Three Important Sums

(1) $1+2+3+\ldots+n=\dfrac{n(n+1)}{2}$.

(2) $1^2+2^2+3^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

(3) $1^3+2^3+3^3+\ldots+n^3=\left\{\dfrac{n(n+1)}{2}\right\}^2$.

Proof: (1) This is an A.P. series with first term a = 1 and C.D. d = 1.

Hence, $S_n=\dfrac{n}{2}\{2\times 1+(n-1)\times 1\}=\dfrac{n(n+1)}{2}$.

(2) Note that this is not an A.P. series. Therefore we can not find this sum using the formula given in 2.3.4.

We have, $r^3-(r-1)^3=3r^2-3r+1 \qquad \ldots (A)$.

On putting $r=1,2,3,\ldots,n$ successively in (A), we get

$$1^3-0^3=3\times 1^2-3\times 1+1$$
$$2^3-1^3=3\times 2^2-3\times 2+1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$n^3-(n-1)^3=3\times n^2-3\times n+1$$

Adding the corresponding sides of the above identities, we find that

$$n^3 - 0^3 = 3\times[1^2 + 2^2 + \cdots + n^2] - 3\times[1+2+\cdots+n] + [1+1+\cdots+1]$$

$$= 3\times[1^2 + 2^2 + \cdots + n^2] - 3\times\frac{n(n+1)}{2} + n.$$

$$\Rightarrow 1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}[n^3 + 3\frac{n(n+1)}{2} - n] = \frac{n(n+1)(2n+1)}{6}$$

(3) This series is also not an A.P. series. Therefore its sum can not be found out by using the formula given in 2.3.4.

We have,

$$(r+1)^2 - (r-1)^2 = 4r.$$

This implies, $r^2(r+1)^2 - r^2(r-1)^2 = 4r^3 \quad \cdots \quad (A).$

Now, on putting r = 1, 2, 3, ... n, successively in (A), we get

$$1^2\times 2^2 - 1^2\times 0^2 = 4\times 1^3$$

$$2^2\times 3^2 - 2^2\times 1^2 = 4\times 2^3$$

$$3^2\times 4^2 - 3^2\times 2^2 = 4\times 3^3$$

....

....

$$n^2\times(n+1)^2 - n^2\times(n-1)^2 = 4\times n^3.$$

Adding both sides we get

$$n^2\times(n+1)^2 - 0 = 4(1^3 + 2^3 + 3^3 + \cdots + n^3).$$

This implies that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} = \left\{\frac{n(n+1)}{2}\right\}^2.$$

5.4 GEOMETRIC PROGRESSION AND SERIES

Consider the succession of numbers 4, 12, 36, 108, …. Here we observe that each term is a constant multiple of 3 of the preceding term except the first term. If in a succession of numbers each term is obtained by multiplying the preceding term by a fixed number except the first, then it is called a **Geometric progression.** The fixed number is called the common ratio (C.R.) of the G.P.

If a G.P. contains finite number of terms, then it is called finite G.P. and if the G.P. contains infinite number of terms, then it is called an infinite G.P.

Examples: (i) 2, 8, 32, 128, … is a G.P. with C.R. equal to 4.

(ii) $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$ is a G.P. with C.R. equal to $\frac{1}{2}$.

Remark: If the first and second terms of a G.P. are t_1 and t_2 respectively, then the C.R. of the G.P. is equal to $\frac{t_2}{t_1}$.

We shall usually denote the first term t_1 of a G.P. by *a* and C.R. by *r*. Then the G.P. may be written as

$$\boxed{a, a\times r, a\times r^2, a\times r^3, \mathbf{L}}$$

Hence the th term of the G.P. is $a\times r^{n-1}$, and it is dented by t_n.

Exercise 5: Find the 9th term of the G.P. –6, 18, –54, 162, – ...

Solution: Here the first term is –6 and C.R. is equal to $\frac{18}{-6} = -3$. Hence,

$t_9 = a\times r^8 = (-6)\times(-3)^8 = -39{,}366$.

Exercise 6: Which term of the G.P. 1, –2, –8, ... is 1024?

Solution: Here $a = 1, r = \frac{-2}{1} = -2$. Suppose th term of the G.P. is 1024.

Then, $a\times r^{n-1} = 1024 \Rightarrow (-2)^{n-1} = 2^{10} = (-2)^{10} \Rightarrow n-1 = 10 \Rightarrow n = 11$.

5.4.1 Properties of a G.P.:

As shown in case of an A.P., the following properties can be seen easily from the definition of G.P.

(1) If each term of a G.P. is multiplied or divided by a non-zero number, the resulting terms are in G.P. with the same C.R. That is, if *a, ar, ar*^{2},... is a G.P. with C.R. = *r*, then *ba, bar, bar*^{2},... is again a G.P. with C.R. = $\frac{bar}{ba} = r$.

(2) If is a G.P. with C.R. =, then the reciprocals $\frac{1}{a}, \frac{1}{ar}, \frac{1}{ar^2}, \mathbf{L}$ is again a G.P. with C.R. = $\frac{1/ar}{1/a} = \frac{1}{r}$.

(3) (i) If $a > 0, r > 1$, then the second term is bigger than the first term, third term is bigger than the second term, and so on. In

this case the G.P. is said to be increasing.

(ii) If $a > 0, 0 < r < 1$, then the G.P. is said to be decreasing.

(iii) If $a < 0, r > 1$, then the G.P. is said to be decreasing.

(iv) If $a < 0, 0 < r < 1$, then the G.P. is said to be increasing.

5.4.2 Geometric Series

If the terms $t_1, t_2, t_3, \ldots, t_n, \ldots$ of a G.P. are added in the given order then $t_1 + t_2 + t_3 + \ldots + t_n + \ldots$ is called a **Geometric Series or G.P. series.**

Examples:

(i) 1+ 2 + 4 + … + 64 is a G.P. series.

(ii) 5 + 15 + 45 + … + 1215 is a G.P. series.

5.4.3 Sum of first n terms of a G.P. Series

Let S_n be the sum of the first n terms of the G.P. $a, ar, ar^2, \ldots$. Then,

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \qquad (1).$$

Multiplying both sides of (1) by r, we get

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n \qquad (2).$$

Subtracting (2) from (1), we find that

$$S_n - rS_n = a - ar^n \Rightarrow (1-r)S_n = a(1-r^n) \Rightarrow S_n = \frac{a(1-r^n)}{1-r}, \quad r \neq 1.$$

Remark: If $a = 1$, then $1 + r + r^2 + \ldots + r^{n-1} = \frac{1-r^n}{1-r}$.

Examples:

(i) $1+2+4+\ldots+64 = 1+2+2^2+\ldots+2^6 = \frac{1\times(1-2^7)}{1-2} = 127$.

(ii) The sum of first 6 terms of the G.P. series: 5+15+45+... is $S_6 = \frac{5\times(1-3^6)}{1-3} = 1820.$

CHECK YOUR PROGRESS

Q 1: State whether the following statements are True (T) or False (F).

(i) 18, 11, 4, -3,…is an A.P. with C.D.= -7. (T / F)

(ii) The n^{th} term of the A.P. 12, 14½, 17,…is ½ (5n + 17). (T / F)

(iii) The multiples of 5 lying between 10 and 100 form an A.P. (T/F)

(iv) 143 is a term of the A.P. 15, 23, 31,… (T / F)

(v) The 7^{th} and 13^{th} term of an A.P. are 35 and 63 respectively. The 19^{th} term is 93. (T / F)

(vi) The nth term of an A.P. is 3n-1. The C.D. is 3. (T / F)

Q 2: Fill up the blanks.

(i) $1^2 + 2^2 + \ldots + 10^2$ = ……..

(ii) $1+3+5+\ldots+(2n-1) = \ldots\ldots.$

Q 3: State whether the following statements are True (T) or False (F).

(i) $0.3+0.03+0.003+\ldots$ is a G.P. series with C.R. = $\frac{1}{10}$. (T / F)

(ii) The 9th term of the G.P. $2, -1, \frac{1}{2}, -\frac{1}{4}, \ldots$ is –192. (T / F)

(iii) 6^{th} term of $1, -\frac{4}{5}, \frac{16}{25}, \ldots$ is $-\frac{32}{3225}$. (T / F)

(iv) $\frac{1}{4\sqrt{2}}$ is a term of the G.P. $\sqrt{2}, 1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \ldots$ (T / F)

(v) If 5^{th} term and 8^{th} term of a G.P. are 16 and 128 respectively, the 10th term is 512. (T / F)

5.5 ARITHMETIC AND GEOMETRIC MEANS

5.5.1. Arithmetic Mean

If *a, b, c* are three numbers such that they are in A.P., then is called the Arithmetic Mean (A.M.) of *a* and *c*. Since are in A.P., we have

$$\boxed{b - a = c - b \Rightarrow b = \frac{a+c}{2}}$$

Thus the A.M. of two numbers and is equal to the average of them. If $a, m_1, m_2, \mathbf{L}, m_n, b$ are in A.P., then $m_1, m_2, \mathbf{L}, m_n$ are said to be n arithmetic means between *a* and *b.*

Now, $a, m_1, m_2, \mathbf{L}, m_n, b$ are in A.P. and clearly there are $n+2$ terms in this A.P. If the C.D. is *d,* then *n+2)* th term is $a+[(n+2)-1]d = a+(n+1)d$,

which is equal to *b*. Thus, $a+(n+1)d = b \Rightarrow d = \dfrac{b-a}{n+1}$.

Using this value of *d*, we get.

$$m_1 = a+d = a+\frac{b-a}{n+1}, \quad \mathbf{L}, \quad m_n = a+nd = a+n\frac{b-a}{n+1}.$$

Exercise 6: If 3, *p, q, r, –17* are in A.P., find *p, q, & r.*

Solution: Let d be the C.D. of the given A.P. Then,

$a+4d = -17 \Rightarrow 3+4d = -17 \Rightarrow d = -5.$

Thus, $p = 3+d = -2, \quad q = 3+2d = -7, \quad r = 3+3d = -12$

5.5.2 Geometric Mean

If *a, b, c* are three numbers such that they are in G.P., then is called the Geometric Mean (G.M.) of *a* and *c*. Since *a, b, c* are in G.P., we have

$$\boxed{\frac{b}{a} = \frac{c}{b} \Rightarrow b^2 = ac \Rightarrow b = \sqrt{ac}}$$

Thus the G.M. of two numbers *a* and *c* is equal to the square root of their product. If $a, m_1, m_2, \mathbf{L}, m_n, b$ are in G.P., then $m_1, m_2, \mathbf{L}, m_n$ are said to be *n* geometric means between *a* and *b*.

Now, $a, m_1, m_2, \mathbf{L}, m_n, b$ are in G.P. and clearly there are $n+2$ terms in this G.P. If the C.R. is '*r*', then (*n*+2) th term is $a \times r^{(n+2)-1} = a \times r^{n+1}$, which is equal to *b.* Thus,

$$a \times r^{n+1} = b \Rightarrow r^{n+1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Using this value of , we get.

$$m_1 = a \times r = a \times \left(\frac{b}{a}\right)^{\frac{1}{n+1}}, \quad \mathbf{L}, \quad m_n = a \times r^n = a \times \left(\frac{b}{a}\right)^{\frac{n}{n+1}}.$$

Exercise 7: Find 5 Geometric means between 3 and 192.

Solution: Let G_1, G_2, G_3, G_4, and G_5 be the 5 G.M. between 3 and 192.

Here a = 3 and the 7[th] term is 192. If *r* is the C.R., then

$$a \times r^6 = 192 \Rightarrow 3 \times r^6 = 192 \Rightarrow r^6 = 64 = 2^6 \Rightarrow r = 2$$

Thus, G_1 = 3 x 2 = 6, G_2 = 3 x 22 = 12, G_3 = 24, G_4 = 48, G_5 = 96.

5.5.3 Relation between A.M. and G.M.

Let *a* and *b* be two real numbers. Then their A.M. and G.M. are $\frac{a+b}{2}$ and $\sqrt{ab}$ respectively. Now,

A.M. - G.M = $\frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{1}{2}(\sqrt{a}-\sqrt{b})^2 \;\; \mathbf{L}\,(1)$

Again, for any two real numbers *a* and *b*,

$$(\sqrt{a}-\sqrt{b})^2 \geq 0 \Rightarrow \frac{1}{2}(\sqrt{a}-\sqrt{b})^2 \geq 0 \quad \mathbf{L}\,(2)$$

From (1) and (2), we get **A.M.** ^{3} **G.M.**

Special Cases:

(i) If $a \neq b$, then $\frac{1}{2}(\sqrt{a}-\sqrt{b})^2 > 0$ and hence **A.M. > G.M.**

(ii) If *a* =*b*, then **A.M. = G.M.**

Exercise 8: If *G* is the G.M. of *a* and *b* , and *p, q* are two A.M.'s between *a* and *b,* prove that $G^2 = (2p-q)(2q-p)$.

Solution: Here *a, p, q, b* are in A.P. and hence $p = \frac{a+q}{2}$ and $q = \frac{p+b}{2}$. This implies that $q = \frac{p+b}{2}$

Again, $G^2 = ab$ and this implies that $G^2 = (2p-q)(2q-p)$

5.6 LET US SUM UP

In this unit, we have learnt about two different patterns of numbers, A.P. and G.P. We derived formulas to find sum of first terms of A.P. and G.P.

series. We have also learnt about Arithmetic and Geometric means and relations between them.

5.7 FURTHER READINGS

1. Khanna V. K., Zameeruddin Qazi & Bhambri S.K.(1995). Business Mathematics, New Delhi, Vikas Publishing House Pvt Ltd. .

2. Hazarika P.L. Business Mathematics, New Delhi. S.Chand & Co.

5.8 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1:

(i) T, (ii) F, (iii) T, (iv) T, (v) F, (vi) T

Ans to Q No 2:

(i) 385, (ii) n^2,

Ans to Q No 3:

(i) T, (ii) F, (iii) T, (iv) T, (v) T

5.9 MODEL QUESTIONS

1. How many terms of the series 12+15+18+... beginning from the first amount to 495?
2. Find four numbers in A.P. if their sum is 20 and sum of their squares is 120.

3. The *p* th term of an A.P. is *a* and the th term is *b*. Prove that the sum of the first (*p*+*q*) terms is $\frac{p+q}{2}\left(a+b+\frac{a-b}{p-q}\right)$.

4. The 5th and 7th term of a G.P. are 81 and 729 respectively. Find the 6th term.

5. Three numbers are in A.P. and their sum is 21. If 1, 2, 15 are added to them in order then the resulting numbers are in G.P. Find the given numbers.

6. If $\frac{a}{b}=\frac{3+2\sqrt{2}}{3-2\sqrt{2}}$, prove that A.M. of *a* and *b* is three times their G.M.

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